Solution to the exam august 2016

Problem 1

a/ $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0 \iff \forall \varepsilon>0\ \exists\delta>0 : \left|\frac{x^2y}{x^2+y^2}\right| < \varepsilon$

if $\|(x,y)\| < \delta$

$$\left|\frac{x^2y}{x^2+y^2}\right| \leq \left|\frac{(x^2+y^2)y}{x^2+y^2}\right| = |y| = |\sqrt{y^2}| \leq |\sqrt{y^2+x^2}| = \|(x,y)\| < \varepsilon$$

for $\delta = \varepsilon$

Alternatively $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} \overset{\text{pol. c}}{=} \lim\limits_{r\to 0} \frac{r^{\cancel{3}}\cos^2(\theta)\sin(\theta)}{\cancel{r^2}}$

$$= \lim_{r\to 0} r \underbrace{\cos^2(\theta)\sin(\theta)}_{\in[-1,1]} = 0$$

b/ $f(\omega,x,y,z) = \omega - x^2y^3z$ is smooth (polynomial). In particular $f\in C^1(\mathbb{R}^4,\mathbb{R})$.

Note that $\lim\limits_{h\to 0} \frac{f(5,2,1+h,-1) - f(5,2,1,-1)}{h} = \frac{\partial f}{\partial y}(5,2,1,-1)$

$$\frac{\partial f}{\partial y}(\omega,x,y,z) = -3x^2y^2z \Rightarrow \underline{\underline{\frac{\partial f}{\partial y}(5,2,1,-1) = 12}}$$

Problem 2

C is parametrized by $c(t) = (t^2, 2t, 4-t)$ $\quad -2\leq t\leq 2$

a/ • The point $(0,0,4)$ lies on the curve : $c(0) = (0,0,4)$

• tangent vector to C : $c'(t) = (2t, 2, -1)$

$c'(0) = (0,2,-1)$ is tangent vector to C at point $(0,0,4)$

$\frac{c'(0)}{\|c'(0)\|} = \underline{\underline{\frac{1}{\sqrt{5}}(0,2,-1)}}$ is tangent vector to C at point $(0,0,4)$ with length 1 b/ Equation for a plane at a point (x_0, y_0, z_0) is given by

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0,$$

where (a,b,c) is a vector orthogonal to the plane.

The plane has to be orthogonal to the curve C, hence $c'(0)$ is a vector orthogonal to the plane at $(0,0,4) = c(0)$

$\Rightarrow \quad (x_0, y_0, z_0) = (0,0,4)$ and $(a,b,c) = c'(0) = (0,2,-1)$

and the plane is given by

$$\underline{\underline{2y - z = 4}}.$$

Problem 3 $\quad f(x,y) = x^4 + x^3 + y^2 + xy$

The second-order Taylor approximation of f at $(x_0, y_0) = (1,2)$ is

$$T_2(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) + \tfrac{1}{2}\big[f_{xx}(1,2)(x-1)^2 + 2f_{xy}(1,2)(x-1)(y-2) + f_{yy}(1,2)(y-2)^2 \big]$$

$f_x(x,y) = 4x^3 + 3x^2 + y \quad \Rightarrow \quad f_x(1,2) = 9$

$f_{xx}(x,y) = 12x^2 + 6x \quad \Rightarrow \quad f_{xx}(1,2) = 18$

$f_y(x,y) = 2y + x \quad \Rightarrow \quad f_y(1,2) = 5$

$f_{yy}(x,y) = 2 \quad \Rightarrow \quad f_{yy}(1,2) = 2$

$f_{xy}(x,y) = 1 \quad \Rightarrow \quad f_{xy}(x,y) = 1$

$$\Rightarrow T_2(x,y) = 8 + 9(x-1) + 5(y-2) + \tfrac{1}{2}\big[18(x-1)^2 + 2(x-1)(y-2) + 2(y-2)^2\big] = \underline{\underline{4 - 11x + 9x^2 + y^2 + xy}}$$

Problem 4 $\quad g(x,y) = e^{xy}$

a/ Find and classify all critical values of g.

(x_0, y_0) critical value $\Leftrightarrow \nabla g(x_0, y_0) = 0$

$$\nabla g(x,y) = \left(y e^{xy}, \; x e^{xy} \right) = (0,0) \overset{e^{xy} \neq 0}{\Longleftrightarrow} (x,y) = (0,0)$$

$\Rightarrow$ there is only one critical value $(x_0, y_0) = (0,0)$

$$Hg(x,y) = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} y^2 e^{xy} & e^{xy} + xy e^{xy} \\ e^{xy} + xye^{xy} & x^2 e^{xy} \end{pmatrix}$$

$\Rightarrow$ $Hg(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\det(Hg(0,0)) = -1 < 0$

$\Rightarrow$ $(0,0)$ is a saddle point

b/ Find all critical points of $g(x,y) = e^{xy}$ unde the constraint that $h(x,y) = 2x^2 + y^2 = 1$

Lagrange Multiplie method : (x_0, y_0) is max / min if $\nabla g(x_0, y_0) = \lambda \nabla h(x_0, y_0)$ for some $\lambda \in \mathbb{R}$.

$\nabla g(x,y) = (ye^{xy}, xe^{xy})$ and $\nabla h(x,y) = (4x, 2y)$

$$\Rightarrow \begin{cases} y e^{xy} = \lambda 4 x & (1) \\ x e^{xy} = \lambda 2 y & (2) \end{cases}$$

If $x = 0 \overset{(1)}{\Longrightarrow} y = 0$, but $(0,0)$ does not satify $h(0,0) = 1$

If $y = 0 \overset{(2)}{\Longrightarrow} x = 0$, //

$\Rightarrow$ $x \neq 0$ and $y \neq 0 \overset{(1),(2)}{\Longrightarrow} \lambda \neq 0$

We can devide (1) by (2) $\Longrightarrow$ $\frac{y}{x} = \frac{2x}{y} \Longleftrightarrow y^2 = 2x^2$

$\overset{h(x,y)=1}{\Longrightarrow}$ $2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm \frac{1}{\sqrt{2}}$

and $x^2 = \frac{1}{2} y^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$

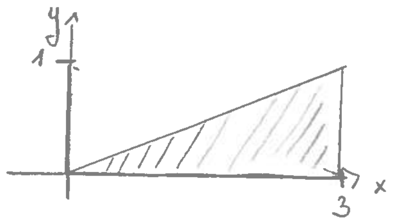
We have 4 critical points $\left(\pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ and $\left(\mp\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$

$g\left(\pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = e^{\frac{1}{2\sqrt{2}}}$ max and $g\left(\mp\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = e^{-\frac{1}{2\sqrt{2}}}$ min

<u>Problem 5</u> Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$ by changing the order of integration

$x \in [3y, 3]$ $\quad y \in [0, 1]$

This domain can be also described by



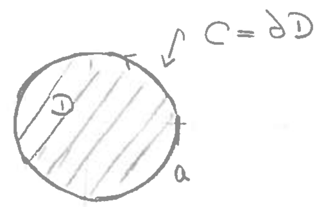
$x \in [0, 3]$ $\quad y \in [0, \frac{x}{3}]$

$$\Rightarrow \int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\, dy\, dx = \int_0^3 \frac{x}{3} e^{x^2}\, dx$$

$$= \frac{1}{6} e^{x^2} \Big|_0^3 = \underline{\underline{\frac{1}{6}\left[e^9 - 1 \right]}}$$

<u>Problem 6</u> Let $F(x,y) = (0, x)$

a/ $$\int_C F \cdot ds \overset{\text{Green}}{=} \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, d(x,y)$$



where $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2 \}$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 \quad \Rightarrow \int_C F \cdot ds = \iint_D 1\, d(x,y) = \text{vol}(D) = \underline{\underline{\pi a^2}}$$

Alternatively: C is parameterized by $c(t) = (x_0 + a\cos(t), y_0 + a\sin(t))$ $t \in [0, 2\pi]$

$$\Rightarrow \int_C F \cdot ds = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt = \int_0^{2\pi} (0, x_0 + a\cos(t)) \cdot (-a\sin(t), a\cos(t))\, dt$$

$$= \int_0^{2\pi} a x_0 \cos(t) + a^2 \cos^2(t)\, dt = \left(a x_0 \sin(t) + a^2 \tfrac{1}{2}\left[t + \cos(t)\sin(t) \right] \right) \Big|_0^{2\pi}$$

$$= \pi a^2$$

b/ Let C be the boundary of the elliptic disk E

$$\overset{\text{Green}}{\Rightarrow} \iint_E 1\, d(x,y) = \int_C F \cdot ds \quad \text{where } F \text{ satisfies } \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

Take for instance $F(x,y) = (0, x)$

$$\Rightarrow \iint_Z 1 \, d(x,y) = \int_C F \cdot ds = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt$$

$$= \int_0^{2\pi} F\big(3(\cos(t)+\sin(t)), 2(\sin(t)-\cos(t))\big) \cdot \big(3(-\sin(t)+\cos(t)), 2(\cos(t)+\sin(t))\big)\, dt$$

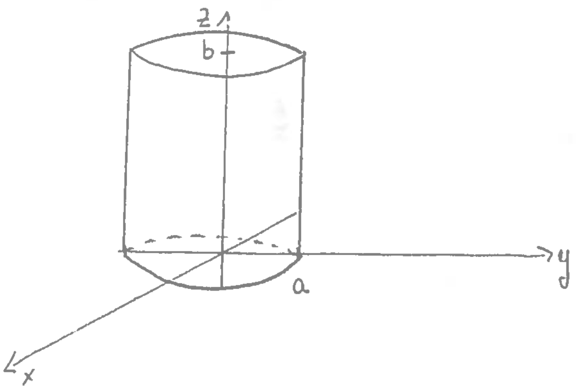
$$= \int_0^{2\pi} \big(0, 3(\cos(t)+\sin(t))\big) \cdot \big(3(-\sin(t)+\cos(t)), 2(\cos(t)+\sin(t))\big)\, dt$$

$$= 6\int_0^{2\pi} (\cos(t)+\sin(t))^2\, dt = 6\int_0^{2\pi} 1 + 2\cos(t)\sin(t)\, dt$$

$$= 12\pi + \sin^2(t))\Big|_0^{2\pi} = \underline{\underline{12\pi}}$$

Problem 7 $\quad F(x,y,z) = (bxy^2, bx^2y, (x^2+y^2)z^2)$

$$W := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le a^2,\ z \in [0,b]\}$$



$$\iint_{\partial W} F \cdot ds \overset{\text{Gauss}}{=} \iiint_W \operatorname{div} F \, ds$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = by^2 + bx^2 + 2(x^2+y^2)z$$

$$\Rightarrow \iint_{\partial W} F\, ds = \iiint_W (2z+b)(x^2+y^2)\, d(x,y,z) \overset{\text{cyl coord}}{=} \int_0^{2\pi}\int_0^a\int_0^b (2z+b)\, r^2 \cdot r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^a 2b^2 r^3\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2} b^2 a^3\, d\theta = \underline{\underline{\pi b^2 a^4}}$$