Solution to Exam 07.06.2016

Problem 1

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- f is not differentiable at $(0,0)$, because f is not continuous at $(0,0)$:

f cont at $(0,0)$ $\Leftrightarrow$ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, but

$$\lim_{(x,x)\to(0,0)} f(x,y) = f(x,x) = \frac{2x^2}{2x^2} = 1 \neq 0 = f(0,0)$$

$\Rightarrow$ f is not cont at $(0,0)$.

- the partial derivatives exist at $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) := \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) := \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

Problem 2

$$c(t) = \left(t, \tfrac{2}{3}\sqrt{2}\, t^{3/2}, \tfrac{1}{2} t^2\right)$$

a/ One has to compute the length of the curve for $t \in [0,10]$:

$$\int_0^{10} \|c'(t)\|\, dt$$

$$c'(t) = (1, \sqrt{2}\, t^{1/2}, t) \Rightarrow \|c'(t)\| = \sqrt{1+2t+t^2} = \sqrt{(1+t)^2} = 1+t$$

$$\Rightarrow \int_0^{10} \|c'(t)\|\, dt = \int_0^{10} 1+t\, dt = 10 + \tfrac{1}{2}100 = 60$$

The fly makes 60 m in 10 seconds

b/ $T(x,y,z) = x^2 + xz + y$ is the temperature.

$$\frac{d}{dt} T(c(t)) = \nabla T(c(t)) \cdot c'(t)$$

$$\nabla T(x,y,z) = (2x+z,\ 1,\ x) \Rightarrow \nabla T\left(t, \tfrac{2}{3}\sqrt{2}\, t^{3/2}, \tfrac{1}{2}t^2\right) = \left(2t + \tfrac{1}{2}t^2, 1, t\right)$$

$$\Rightarrow \frac{d}{dt} T(c(t)) = \left(2t + \tfrac{1}{2}t^2, 1, t\right) \cdot \left(1, \sqrt{2}\, t^{1/2}, t\right) = 2t + \tfrac{1}{2}t^2 + \sqrt{2}\, t^{1/2} + t^2$$

$$\Rightarrow \frac{d}{dt} T(c(t))\Big|_{t=1} = 2 + \tfrac{1}{2} + \sqrt{2} + 1 = 3{,}5 + \sqrt{2}$$

Problem 3

a/ $f(x,y) = 2x^2 + 4xy + y^4$. Find and classify all critical points.

- Critical points: (x,y) crit. point $\Leftrightarrow$ $\nabla f(x,y) = (0,0)$

$$\nabla f(x,y) = (4x + 4y,\ 4x + 4y^3) = (0,0)$$

$\Leftrightarrow$ $x + y = 0$ and $x + y^3 = 0$

$\Leftrightarrow$ $x = -y$ and $-y + y^3 = 0$

$\Leftrightarrow$ $x = -y$ and $y(y^2 - 1) = 0$

$\Rightarrow$ $y = 0$ $(\Rightarrow x = 0)$ OR $y = \pm 1$ $(\Rightarrow x = \mp 1)$

$\Rightarrow$ 3 crit points $(0,0)$, $(1,-1)$, $(-1,1)$

- $$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 12y^2 \end{pmatrix}$$

$(0,0)$: $f_{xx}(0,0) = 4 > 0$

$$\det Hf(0,0) = \det \begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix} = -16 < 0 \quad \Rightarrow \text{saddle point}$$

$\underline{(1,-1)}$: $f_{xx}(1,-1) = 4 > 0$

$$\det(Hf(1,-1)) = \det\begin{pmatrix} 4 & 4 \\ 4 & 12 \end{pmatrix} = 48-16 = 32 > 0 \quad \Rightarrow \text{ local min}$$

$\underline{(-1,1)}$: $f_{xx}(-1,1) = 4 > 0$

$$\det(Hf(-1,1)) = \det\begin{pmatrix} 4 & 4 \\ 4 & 12 \end{pmatrix} = 32 > 0 \quad \Rightarrow \text{ local min.}$$

b/ Find max/min of $f(x,y) = (x-y)^2$ under the constraint $g(x,y) = x^2+y^2 = 2$:

Using Lagrange - Multiplier Method:

(x,y) is crit. point if $\exists \lambda \in \mathbb{R}$: $\nabla f(x,y) = \lambda \nabla g(x,y)$

$\nabla f(x,y) = 2(x-y, -(x-y))$ $\qquad \nabla g(x,y) = 2(x,y)$

$$\nabla f(x,y) = \lambda \nabla g(x,y) \iff \left\{\begin{array}{l} x-y = \lambda x \\ -(x-y) = \lambda y \end{array}\right| + \qquad (*)$$

$\Rightarrow \quad 0 = \lambda x + \lambda y \quad \Rightarrow \quad \lambda x = -\lambda y$

$\Rightarrow \quad \lambda = 0 \quad$ OR $\quad x = -y$

- $\lambda = 0 \overset{(*)}{\Rightarrow} x = y$

  $g(x,y) = x^2+y^2 = 2 \Rightarrow 2x^2 = 2 \Rightarrow x = \pm 1$ and $y = \pm 1$

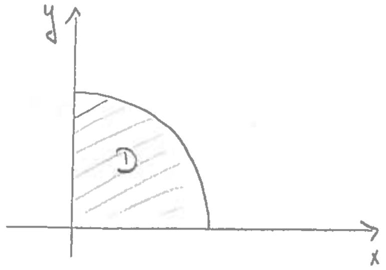
- $\lambda \neq 0$ and $x = -y$

  $g(x,y) = x^2+y^2 = 2 \Rightarrow 2x^2 = 2 \Rightarrow x = \pm 1$ and $y = \mp 1$

$\Rightarrow$ 4 crit. points $(1,1), (-1,-1), (1,-1), (-1,1)$

$$\underbrace{f(1,1) = 0 \qquad f(-1,-1) = 0}_{\text{min}} \qquad \underbrace{f(1,-1) = 4 \qquad f(-1,1) = 4}_{\text{max}}$$

Problem 4



$$\int_D f(x,y)\, d(x,y) = \int_0^1 \int_0^{\frac{\pi}{2}} r^2 \cdot r\, d\theta dr$$

$$= \frac{\pi}{2} \int_0^1 r^3 dr = \frac{\pi}{8}$$

Problem 5 $\quad F(x,y,z) = (2xy, x^2, z)$

Note that $F = \nabla f$ for $f(x,y,z) = y x^2 + \frac{1}{2} z^2$

i/ $c : [0, 2\pi] \to \mathbb{R}^3$, $c(t) = (\cos t, \sin 2t, \sin^2 t)$ is a closed curve

$F = \nabla f$
$$\Rightarrow \int_C F \, ds = 0$$

ii/ Let C be any curve with startpoint $(0,0,0)$ and endpoint $(1,-2,\sqrt{2})$

$$\Rightarrow \int_C F \, ds = \int_C \nabla f \, ds = f(1,-2,\sqrt{2}) - f(0,0,0) = -2 + 1 = -1$$

Problem 6 $\quad f(x,y) = e^{kx}\cos(ky)$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = k^2 e^{kx}\cos(ky) - k^2 e^{kx}\cos(ky) = 0 \quad \Rightarrow f \text{ is harmonic}$$

Problem 7 $\quad f : \mathbb{R}^3 \to \mathbb{R} \quad C^2, \quad F\ \mathbb{R}^3 \to \mathbb{R}^3 \quad C^2$

a/ $$\operatorname{div}(fF) = \operatorname{div}(fF_1, fF_2, fF_3) = \frac{\partial fF_1}{\partial x} + \frac{\partial fF_2}{\partial y} + \frac{\partial fF_3}{\partial z}$$

$$\overset{\text{prod.}}{\underset{\text{rule}}{=}} \frac{\partial f}{\partial x} F_1 + f \frac{\partial F_1}{\partial x} + \frac{\partial f}{\partial y} F_2 + f \frac{\partial F_2}{\partial y} + \frac{\partial f}{\partial z} F_3 + f \frac{\partial F_3}{\partial z}$$

$$= \nabla f \cdot F + f \operatorname{div} F \qquad (*)$$

Gauss' Theorem : $\iiint\limits_{W} \operatorname{div} G \, dV = \iint\limits_{\partial W} G \cdot n \, dS$

$G := f F$

$$\Rightarrow \quad \underline{\iint\limits_{\partial W} f F \cdot n \, dS} = \iiint\limits_{W} \operatorname{div}(fF) \, dV \overset{(*)}{=} \underline{\iiint\limits_{W} f \operatorname{div} F \, dV + \iiint\limits_{W} \nabla f \, F \, dV}$$

b/ $\begin{cases} \nabla^2 f(x,y,z) = 0 , & (x,y,z) \in W \\ f(x,y,z) = 0 & (x,y,z) \in \partial W \end{cases}$

To show: $f \equiv 0$ in W

Set $F = \nabla f$

a/

$$\Rightarrow \quad \underbrace{\iint\limits_{\partial W} f \nabla f \cdot n \, dS}_{= 0 \; (f = 0 \text{ on } \partial W)} = \underbrace{\iiint\limits_{W} f \underbrace{\operatorname{div} \nabla f}_{\nabla^2 f = 0} \, dV}_{= 0} + \iiint\limits_{W} \nabla f \, \nabla f \, dV$$

$$\Rightarrow \quad \iiint\limits_{W} \underbrace{\| \nabla f \|^2}_{\geq 0} dV = 0 \quad \Rightarrow \quad \nabla f = 0 \quad \text{in } W$$

Assume there exist a point $(x,y,z) \in W$ with $f(x,y,z) \neq 0$ (A) and let

C be any smooth curve connecting (x,y,z) to a boundary point $(\bar{x}, \bar{y}, \bar{z}) \in \partial W$

$$\Rightarrow \quad f(x,y,z) - \underbrace{f(\bar{x}, \bar{y}, \bar{z})}_{= 0} = \int\limits_{C} \underbrace{\nabla f}_{= 0} \, dS = 0$$

$\Rightarrow \quad f(x,y,z) = 0$ ↯ to (A)

$\Rightarrow \quad f \equiv 0$ in W