Exercises
Vector Calculus (MA1103)
Exercise 5

Exercise 1 (2.6: 2)

Compute the directional derivatives of the following functions at the indicated points in the given direction:

a) $f(x, y) = x + 2xy - 3y^2$ at $(x_0, y_0) = (1, 2)$ in direction $\mathbf{v} = \frac{1}{5}(3, 4)$

b) $f(x, y) = \log(\sqrt{x^2 + y^2})$ at $(x_0, y_0) = (1, 0)$ in direction $\mathbf{v} = \frac{1}{\sqrt{5}}(2, 1)$

c) $f(x, y) = e^x \cos(\pi y)$ at $(x_0, y_0) = (0, -1)$ in direction $\mathbf{v} = -\frac{1}{\sqrt{5}}(1, -2)$

d) $f(x, y) = xy^2 + x^3 y$ at $(x_0, y_0) = (4, -2)$ in direction $\mathbf{v} = \frac{1}{\sqrt{10}}(1, 3)$

Exercise 2 (2.6: 4)

You are walking on the graph of $f(x, y) = y\cos(\pi x) - x\cos(\pi y) + 10$, standing at the point $(2, 1, 13)$. Find an x, y-direction you should walk in to stay at the same level.

Exercise 3 (2.6: 8a))

Find the plane tangent to the surface $x^2 + 2y^2 + 3xz = 10$ at the point $(1, 2, \frac{1}{3})$.

Exercise 4 (2.6: 19)

You are standing on the graph of $f(x, y) = 100 - 2x^2 - 3y^2$ at the point $(2, 3, 65)$.

a) What are the xy coordinates of the highest point on the graph?

b) Show that the gradient of f is the zero vector at the point found in a).

Exercise 5 (2.6: 26)

Suppose that a mountain has the shape of an elliptic paraboloid $z = c - ax^2 - by^2$, where a, b and c are positive constants, x and y are the east–west and north–south map coordinates, and z is the altitude above sea level (x, y, z are all measured in meters). At the point $(1, 1)$, in what direction is the altitude increasing most rapidly? If a marble were released at $(1, 1)$, in what direction would it begin to roll?

Exercise 6 (3.1: 2)

Compute the second partial derivatives $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ for

$$f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad (x, y, z) \mapsto e^z + \frac{1}{x} + xe^{-y}$$

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Exercise 7 (3.1: 7)

Find all second partial derivatives of the following functions at the point $\mathbf{x}_0$

a) $f(x,y) = \sin(xy)$ at $\mathbf{x}_0 = (\pi, 1)$

b) $f(x,y) = xy^8 + x^2 + y^4$ at $\mathbf{x}_0 = (2, -1)$

c) $f(x,y,z) = e^{xyz}$ at $\mathbf{x}_0 = (0,0,0)$

Exercise 8 (3.1: 10)

The heat conduction equation is $u_t = ku_{xx}$. Determine whether $u(x,t) = e^{-kt}\sin(x)$ is a solution.

Exercise 9 (3.1: 11)

Show that the following functions satisfy the one-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

a) $f(x,t) = \sin(x - ct)$

b) $f(x,t) = \sin(x)\sin(ct)$

c) $f(x,t) = (x - ct)^6 + (x + ct)^6$

Exercise 10 (3.1: 23)

Let $f \in C^2(\mathbb{R}^2, \mathbb{R})$ and $c \in C^2(\mathbb{R}, \mathbb{R}^2)$. Write a formula for the second derivative $(\frac{d^2}{dt^2})(f \circ c)(t)$ using the chain rule twice.

The exercise can be also found (under the given number in brackets) in the book *Vector Calculus* by J. E. Marsden and A. Tromba.