Exercises
Vector Calculus (MA1103)
Exercise 3

Exercise 1 (2.2: 6)

Let

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6}, & \text{if} \quad (x,y) \neq (0,0), \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

a) Compute the limit as $(x,y) \to (0,0)$ of f along the path $x = 0$.

b) Compute the limit as $(x,y) \to (0,0)$ of f along the path $x = y^3$.

c) Show that f is not continuous at $(0,0)$.

Exercise 2 (2.2: 7)

Let $f(x,y,z) = \frac{e^{x+y}}{1+z^2}$. Compute $\lim_{h\to 0} \frac{f(1,2+h,3)-f(1,2,3)}{h}$.

Exercise 3 (2.2: 11)

Compute the following limits if they exist:

a) $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$

b) $\lim_{(x,y,z)\to(0,0,0)} \frac{\sin(xyz)}{xyz}$

c) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+3y^2}{x+1}$

Exercise 4 (2.2: 25) a) Can $f(x,y) = \frac{\sin(x+y)}{x+y}$ be made continuous by suitably defining it at $(0,0)$?

b) Can $\frac{xy}{x^2+y^2}$ be made continuous by suitably defining it at $(0,0)$?

c) Prove that $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto ye^x + \sin(x) + (xy)^4$ is continuous.

Exercise 5 (2.2: 26)

Show that $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0$

Exercise 6 (2.2: 33)

Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ satisfy $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|^\alpha$ for all $\mathbf{x}, \mathbf{y}$ in A and positive constants K and α. Show that f is continuous. (Such functions are called *Hölder-continuous*. In the special case, when $\alpha = 1$, they are called *Lipschitz-continuous*)

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Exercise 7 (**A** 2.2: 34)

Show that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if the inverse image $f^{-1}(U) \subset \mathbb{R}^n$ of every open set $U \subset \mathbb{R}^m$ is open.

Exercise 8 (2.3: 1)

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if

a) $f(x, y) = xy$

b) $f(x, y) = e^{xy}$

c) $f(x, y) = x\cos(x)\cos(y)$

d) $f(x, y) = (x^2 + y^2)\log(x^2 + y^2)$

Exercise 9 (2.3: 5)

Find the equation of the plane tangent to the surface $z = x^2 + y^3$ at $(3, 1, 10)$.

Exercise 10 (2.3: 9)

Compute the matrix of partial derivatives of the following functions:

a) $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (x, y)$

b) $f : \mathbb{R}^2 \to \mathbb{R}^3, f(x, y) = (xe^y + \cos(y), x, x + e^y)$

c) $f : \mathbb{R}^3 \to \mathbb{R}^2, f(x, y, z) = (x + e^z + y, yx^2)$

d) $f : \mathbb{R}^2 \to \mathbb{R}^3, f(x, y) = (xye^{xy}, x\sin(y), 5xy^2)$

A: This exercise is more theoretical (and might therefore be more difficult).

The exercise can be also found (under the given number in brackets) in the book *Vector Calculus* by J. E. Marsden and A. Tromba.