On Taylor polynomials and Taylor serieses

Let $f \in C^{\infty}(\mathbb{R})$

The Taylor polynomial at x_0 is : $T_n f(x) = \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$

and $f(x) = T_n f(x) + R_n(x)$ where R_n is the „error" given by

$$R_n(x) = \int_{x_0}^{x} \frac{(x-z)^n}{n!} f^{(n+1)}(z)\, dz$$

NOTE $\lim_{x \to x_0} \frac{R_n(x)}{(x-x_0)^n} = 0$

$f^{(n+1)}$ cont $\Rightarrow$ bounded on the compact intervall $[x_0, x]$

$$\Rightarrow |f^{(n+1)}(z)| \leq M \quad \forall z \in [x_0, x]$$

$$\Rightarrow |R_n(x)| \leq \int_{x_0}^{x} \left| \frac{(x-z)^n}{n!} f^{(n+1)}(z) \right| dz$$

$$\overset{z \in [x_0, x]}{\leq} \int_{x_0}^{x} \frac{|(x-x_0)|^n}{n!} |f^{(n+1)}(z)|\, dz \leq \frac{M}{n!} |x-x_0|^{n+1}$$

$$\Rightarrow \frac{|R_n(x)|}{|x-x_0|^n} \leq \frac{M}{n!} |x-x_0| \longrightarrow 0 \quad \text{if } x \to x_0$$

Taylor series

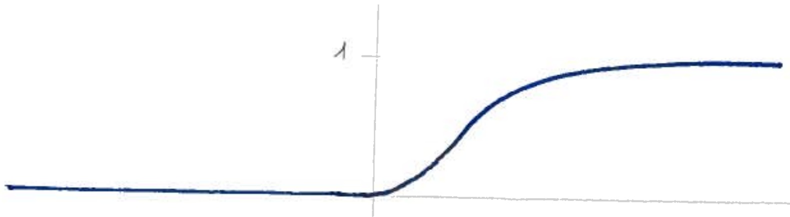
$$T f(x) = \sum_{k=1}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$$

$\longrightarrow$ Does it converge? For which x does it converge?

$\longrightarrow$ If it converges in a nbh of x_0 to which function?

Exampel

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & \text{else} \end{cases} \Rightarrow f \in C^\infty(\mathbb{R})$$



One may check that $f^{(k)}(0) = 0$ for all $k \geq 0$

$\Rightarrow$ The Taylor polynomial of f at point $x_0 = 0$ is

$$T_n f(x) = \sum_{k=0}^{n} \frac{1}{k!} \underbrace{f^{(k)}(0)}_{=0} x^k = 0 \qquad \forall n \in \mathbb{N}$$

$\Rightarrow$ The Taylor series is

$$T f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) x^k = 0$$

$\Rightarrow$ The Taylor series converges! It converges for all $x \in \mathbb{R}$!

$\Rightarrow$ It converges NOT to f (not in any nbh of $x_0 = 0$)!