12 ØVING

Med vanlige forbehold,
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Løsninger/kommentarer til oppgavene

7.6.6

Se ex 4, s 238 for varmefluksen $\vec{J} = -k\vec{\nabla}T$ der temperaturen $T = T(x,y,z)$. (Varmen strømmer i retningen der temperaturen faller mest.)

Her $T(x,y,z) = x$, $\underline{\vec{\nabla}T = \vec{i}}$ og $\underline{\vec{J} = -k\vec{i}}$

$$\iint_S (-k\vec{i})\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} (-k\vec{i})\cdot(\vec{T}_\phi \times \vec{T}_\theta)\,d\phi\,d\theta$$

$$\vec{T}_\phi \times \vec{T}_\theta = (\sin^2\phi\cos\theta, \cdots)$$ $\vec{T}_\phi \times \vec{T}_\theta$ utadrettet normal

Se utregning s. 401.

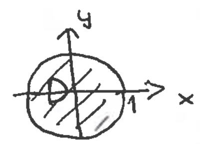
Altså

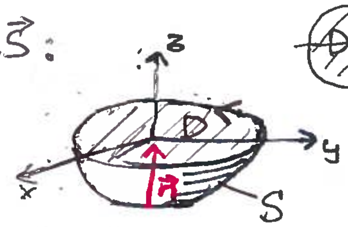
$$\iint_S (-k\vec{i})\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} -k\sin^2\phi\cos\theta\,d\phi\,d\theta$$

$$= -k\int_0^{\pi}\sin^2\phi\,d\phi\,\underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{0} = \underline{\underline{0}}$$

Det strømmer like mye varme inn og ut av enhetssfæren, noe som ikke er overraskende da antipodale punkter har motsatt temperatur.

7.6.9





Direkte utregning av $\iint_S (\nabla\times\vec{F})\cdot d\vec{S}$:

Gitt $S: x^2+y^2+3z^2 = 1,\ z \le 0$

$\vec{F} = (y, -x, zx^3y^2)$

Altså

$$S:\ z = \underbrace{-\frac{1}{\sqrt{3}}\sqrt{1-x^2-y^2}}_{g(x,y)}\ ;\quad x^2+y^2 \le 1\ (D)$$

$$g_x = -\frac{x}{3z},\quad g_y = -\frac{y}{3z}$$

og $\vec{\nabla} \times \vec{F} = (2yzx^3, -3x^2y^2z, -2)$ slik at

2,c, s.411 (Kan du utlede?) $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} \overset{*}{=} \iint_D \left[2yzx^3\left(\frac{x}{3z}\right) - 3x^2y^2z\left(\frac{y}{3z}\right) - 2\right] dx\,dy$

$$= \iint_D \left[\tfrac{2}{3} y x^4 - x^2 y^3 - 2\right] dx\,dy$$

$x = r\cos\theta, y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 \left[\tfrac{2}{3} r^6 \sin\theta\cos^4\theta - r^6 \cos^2\theta \underbrace{\sin^3\theta}_{(1-\cos^2\theta)\sin\theta} - 2r\right] dr\,d\theta$$

$$= 0 - 0 - \int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = -2\pi \cdot 1 = \underline{\underline{-2\pi}}$$

Dette var ganske tungt! Ved å bruke Stokes kommer vi mye lettere til svaret.

<u>Alt1</u> (Stokes rett fram): $\vec{F} = (y, -x)$ når $z = 0$

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_{\partial S = C} \vec{F} \cdot d\vec{s} \underset{\substack{\vec{c}(t) = (\cos t, \sin t) \\ 0 \le t \le 2\pi}}{=} \int_0^{2\pi} (\sin t, -\cos t) \cdot \vec{c}\,'(t)\,dt$$

Orientering S og ∂S samsvarer

$$= \int_0^{2\pi} \underbrace{(\sin t, -\cos t)(-\sin t, \cos t)}_{-(\sin^2 t + \cos^2 t)} dt = \underline{\underline{-2\pi}}$$

<u>Alt 2</u> (Parametriserer ikke ∂S, men bruker Green):

$$\int_{C = \partial S} \vec{F} \cdot d\vec{s} = \int_C y\,dx - x\,dy \underset{\text{Green}}{=} \iint_D \frac{\partial(-x)}{\partial x} - \frac{\partial y}{\partial y} = -2 \iint_D 1\,dA = \underline{\underline{-2\pi}}$$

omslutter $x^2 + y^2 \le 1$ (D)

NB! <u>Alt 3</u> ($\partial S = \partial D$, $D: x^2 + y^2 \le 1$)

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_{\partial S = \partial D} \vec{F} \cdot d\vec{s} = \iint_D \vec{\nabla} \times \vec{F} \cdot \vec{k}\,dA =$$

$$\underset{z=0}{=} \iint_D (0, 0, -2) \cdot (0, 0, 1) = \underline{\underline{-2\pi}}$$

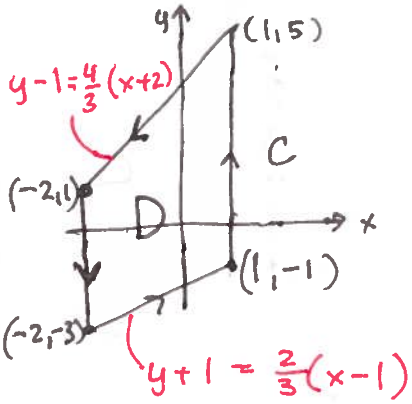
<u>Alt 4</u>

$$0 = \iiint_W \operatorname{div}(\vec{\nabla} \times \vec{F})\,dV = \iint_{\partial W = S \cup D} \vec{\nabla} \times \vec{F} \cdot d\vec{S} = -\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} + (-2\pi)$$

Normal <u>UT</u> i DT

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \underline{\underline{-2\pi}}$$

8.1.7



Har

$$\int_C (2x+y)\,dx + (xy^2)\,dx$$

$$\underset{\text{Green}}{=} \iint_D (y^2-2x)\,dA$$

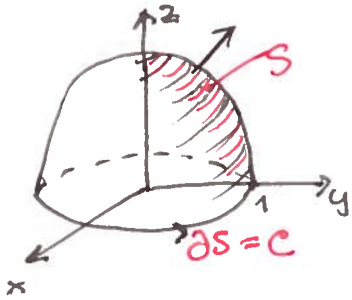
$$= \int_{-2}^{1}\int_{\frac{2}{3}x-\frac{5}{3}}^{\frac{4}{3}x+\frac{11}{3}} (y^2-2x)\,dy\,dx$$

$$= \int_{-2}^{1} \frac{1}{3}\left[\left(\frac{4}{3}x+\frac{11}{3}\right)^3-\left(\frac{2}{3}x-\frac{5}{3}\right)^3 - 2x\left(\frac{4}{3}x+\frac{11}{3}-\frac{2}{3}x+\frac{5}{3}\right)dx\right.$$

$$= \frac{1}{8}\left[\left(\frac{4}{3}x+\frac{11}{3}\right)^4\cdot\frac{1}{2}-\left(\frac{2}{3}x-\frac{5}{3}\right)^4\right]_{-2}^{1} - \left[\frac{4x^3}{3\cdot 3}+\frac{16}{3}x^2\right]_{-2}^{1}$$

$$= \frac{1}{8}\left[\left(5^4\cdot\frac{1}{2}-1\right)-\left(\frac{1}{2}-81\right)\right] - \frac{52}{9}+\frac{160}{9} = \frac{392}{8}+\frac{108}{9} = 49+12 = \underline{\underline{61}}$$

8.2.5



S: $z = 1-x^2-y^2$, $(2x,2y,1)\perp S$ (peker ut)

$\bar{F} = (z, x, 2xz+2xy)$. Ser først på $\int_{\partial S} \bar{F}\cdot d\bar{s}$:

$\partial S = C$: $x=\cos t$, $y=\sin t$, $z=0$; $0\le t\le 2\pi$

For $z=0$ er $\bar{F} = (0, x, 2xy)$. Altså:

H.S. Stokes $\int_{\partial S}\bar{F}\cdot d\bar{s} = \int_C 0\,dx + x\,dy + 2xy\,dz = \int_0^{2\pi}\cos^2 t\,dt = \underline{\underline{\pi}}$

V.S. Stokes Har $\text{curl}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & 2zx+2xy \end{vmatrix} = (2x, -(2z+2y-1), 1) = (2x, 1-2y(1-x^2-y^2), 1)$

For flateintegralet (2.c. s. 411)

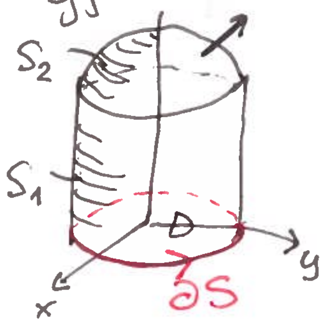
$$\iint_S \text{curl}\,\bar{F}\cdot d\bar{S} = \iint_{D_1} (2x, -(2z+2y-1), 1)\frac{(2x,2y,1)}{\cancel{\sqrt{(2x)^2+(2y)^2+1}}}\cancel{\sqrt{}}\,dxdy$$

$$= \iint_{D_1} (4x^2 - 4y(1-x^2-y^2) - 4y^2 + 2y + 1)\,dxdy$$

$$= \iint_{D_1} 1\,dxdy = \underline{\underline{\pi}}$$ ved symmetri eller utregning i polark.

#8.2.13

S er her en stykkevis glatt flate (Hint; Stokes gjelder for S) Altså



$$\iint_S (\nabla \times \underline{F}) \cdot d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{s} \overset{*}{=} \int_{\partial S} x\,dx + y\,dy = \underline{\underline{0}}$$

$$\underline{F}(x,y,z) = (zx + z^2 y + x,\ z^3 yx + y,\ z^4 x^2)$$

$$* \quad \underline{F}(x,y,0) = (x,\ y,\ 0)$$

At det siste integralet er 0 kan vi innse på mange måter:

- Ved Green $\int_{\partial S} x\,dx + y\,dy = \iint_D \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) dA = \iint_D 0\,dA = 0$
- Ved at feltet er et gradientfelt da $(x,y) = \text{grad}\,\frac{x^2+y^2}{2}$ (og ∂S en lukket kurve).
- Ved å parametrisere ∂S: $\underline{c}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

Alternativt kunne vi brukt divergensteoremet som i Alt4 s.2:

$$0 = \iiint_W \operatorname{div}(\vec{\nabla} \times \vec{F})\,dV = \iint_{S \cup D \text{ Normal UT}} \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} + \underbrace{\iint_D \vec{\nabla} \times \vec{F} \cdot (-\vec{k})\,dA}_{0}$$

#8.3.5

Vi skal vise at to potensialfunksjoner til samme (glatte) felt $\underline{F}$ i $\mathbb{R}^3$ er like på en konstant nær. Fasiten gir hint til en elegant løsning: Vi ser på $d = f - g$ der f og g er to potensial-funksjoner til $\underline{F}$; $\nabla d = \nabla f - \nabla g = \underline{0}$. Fiksér videre $P_0 (\in \mathbb{R}^3)$, La P være vilkårlig, og la $\underline{c}$ være en C^1-kurve fra P_0 til P. Har da, T 3 s. 367, $d(P) - d(P_0) = \int_{\underline{c}} \nabla d \cdot d\underline{s} = 0$, dvs. $d(P) = d(P_0)$.

Altså $\underline{f(P) - g(P)} = f(P_0) - g(P_0) = \underline{\text{konstant}}$, og fram me!

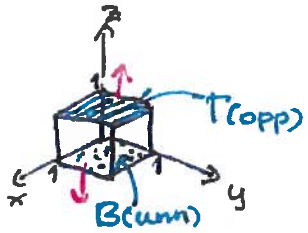
#8.3.11

Vi vet at $\text{curl}(\vec{\nabla} f) = \vec{0}$ for f C^2. (T I s. 252)

Her er $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y & z \end{vmatrix} = (0,0,-x) \neq \vec{0}$

($\vec{F}$ er definert i hele $\mathbb{R}^3$.)

#8.4.7



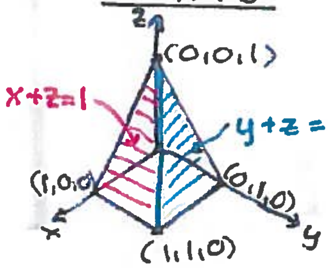
$$\iint_B (x\vec{i} + y\vec{j}) \cdot (-\vec{k})\, dA = 0$$

$$\iint_T (x\vec{i} + y\vec{j} + 1\vec{k}) \cdot \vec{k}\, dA = 1$$

Ved symmetri blir flateintegralet $\underline{3}$.

Kontroll: $\text{div}(x\vec{i} + y\vec{j} + z\vec{k}) = 3$ $\quad \iiint_{\text{Enhetskule}} \text{div}\,\vec{F}\, dV = 3\iiint_E dV = 3 \cdot 1 = \underline{\underline{3}}$

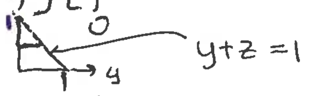
#8.4.13



$\text{div}\,\vec{F} = 2xy + 6yz + 18zx$ $\quad \iint_{\partial W} \vec{F} \cdot d\vec{S} = ?$ Ved Gauss

$\iiint_W \text{div}\,\vec{F}\, dV = ?$ Får enklest uttrykk for volum-integralet ved å se W som volumet under den røde (eller grønne) trekanten!

$$\iiint_W \text{div}\,\vec{F}\, dV = \int\!\!\int \left[\int_0^{1-z} (2xy + 6yz + 18zx)\, dx \right] dy\, dz$$



$$= \int_0^1 \int_0^{1-z} y(1-z)^2 + 6yz(1-z) + 9z(1-z)^2\, dy\, dz$$

$$= \int_0^1 \tfrac{1}{2}(1-z)^4 + 12z(1-z)^3\, dz$$

$$= \left[-\tfrac{1}{10}(1-z)^5 + 6z^2 - 12z^3 + 9z^4 - \tfrac{12}{5}z^5 \right]_0^1$$

$$= 6 - 12 + 9 - \tfrac{12}{5} + \tfrac{1}{10} = \underline{\underline{\tfrac{7}{10}}}$$