1 given the non-degenerate conic

$$3x^2-6x+4y^2=0$$

we have that the discriminant is

$$0-4\cdot3\cdot4=-48<0,$$

so it is an $\boxed{\text{ellipse}}$.

We can write the equation of the ellipse in the following way

$$3x^2-6\cancel{x}+4y^2=0 \Rightarrow 4y^2=-3x^2+6x \Rightarrow$$

$$\Rightarrow 4y^2=-3(x^2-2x) \Rightarrow 4y^2=-3(x^2-2x+1-1) \Rightarrow$$

$$\Rightarrow 4y^2=-3((x-1)^2-1) \Rightarrow \boxed{y^2=-\frac{3}{4}((x-1)^2-1)}$$

Then the center is $\boxed{\bar{x}=1}$ and we have that

$$\varepsilon^2-1=-\frac{3}{4} \Rightarrow \varepsilon^2=\frac{1}{4} \Rightarrow \boxed{\varepsilon=\frac{1}{2}}$$

2 $\vec{r}(t)=(e^t-t,\ 4e^{t/2})$ for $t\geq0$

$$\vec{v}(t)=\vec{r}\,'(t)=(e^t-1,\ 2e^{t/2}) \qquad t\geq0$$

$$v(t)=\|\vec{v}(t)\|=\sqrt{(e^t-1)^2+(2e^{t/2})^2}=\sqrt{e^{2t}-2e^t+1+4e^t}$$

$$=\sqrt{e^{2t}+2e^t+1}=\sqrt{(e^t+1)^2}=e^t+1$$

$$\vec{a}(t)=\vec{v}\,'(t)=(e^t,\ e^{t/2})$$

$$a(t)=\frac{\vec{v}(t)}{v(t)}\cdot\vec{a}(t)=e^t$$

Arc length

$$L = \int_0^1 v(t)\, dt = \int_0^1 e^t + 1\, dt = e^t + t \Big]_0^1 = \boxed{e^1}$$

3 a) We compute the roots of the characteristic polynomial of $2y'' + 2y' + y = 0$ that is

$p(r) = 2r^2 + 2r + 1 = 0$, with roots

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{-2 \pm \sqrt{-4}}{4} =$$

$$\boxed{r = -\frac{1}{2} \pm \frac{1}{2} i}$$

Then the solutions of the homogeneous differential equation are.

$$\boxed{y = e^{-\frac{1}{2}x} \cdot \left(A \cdot \cos \tfrac{1}{2}x + B \cdot \sin \tfrac{1}{2}x\right)}$$

for $A, B \in \mathbb{R}$.

b) To find the solutions of

$$2y'' + 2y' + y = x^2 + x + 1$$

we need to find a particular solution y_p, that is a polynomial of degree 2.

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Then $2y_p'' + 2y_p' + y_p = x^2 + x + 1$ so

$$2 \cdot 2A + 2(2Ax + B) + Ax^2 + Bx + C = x^2 + x + 1$$

$$Ax^2 + (4A + B) \cdot x + (4A + 2B + C) = x^2 + x + 1$$

That yields to the system

$$\left.\begin{array}{l} A = 1 \\ 4A + B = 1 \\ 4A + 2B + C = 1 \end{array}\right\} \Rightarrow \begin{array}{l} A = 1 \\ B = -3 \\ C = 3 \end{array}$$

so the solutions of the differential equation are

$$y = \underbrace{x^2 - 3x + 3}_{y_p} + \underbrace{e^{-\frac{1}{2}x}\left(A\cos\tfrac{1}{2}x + B\sin\tfrac{1}{2}x\right)}_{y_h}$$

c) $y = \sum_{n=0}^{\infty} a_n x^n$ satisfying $y'' + 2xy' + 2y = 0$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} \cdot x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+1) \cdot n\, a_{n+1} \cdot x^{n-1} = \sum_{n=1}^{\infty} (n+1) \cdot n\, a_{n+1}\, x^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n$$

Then

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n + 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} \cdot x^n + 2 \sum_{n=0}^{\infty} a_n x^n$$

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \cdot x^n + 2 \sum_{n=1}^{\infty} n \cdot a_n\, x^n + 2 \sum_{n=0}^{\infty} a_n x^n$$

$$0 = \sum_{n=1}^{\infty} \left((n+2)(n+1) a_{n+2} + 2n\, a_n + 2a_n \right) x^n + a_2 + 2a_0$$

this implies that
$$\begin{cases} a_2 + 2a_0 = 0 \\ \text{and} \\ (n+2)(n+1) a_{n+2} + 2(n+1) a_n = 0 \\ \forall n \geq 1 \end{cases}$$

But $y(0) = \boxed{a_0 = 1}$ and $y'(0) = \boxed{a_1 = 0}$

Then $\boxed{a_2 = -2}$. From the equations follows

$$\boxed{a_{n+2} = \frac{-2}{n+2} a_n}$$

Observe that since $a_1 = 0$, then

$a_3 = 0$, $a_5 = 0$, ... so all the odd n

we have that $a_n = 0$, so we can write

$\boxed{a_{2n+1} = 0}$ $\forall n = 0, 1, 2, \dots$

Then we need to compute a_{2n}, but then

$$\boxed{a_{2n+2} = \frac{-2}{2n+2} a_{2n} = -\frac{1}{n+1} a_{2n}}$$

Since $a_2 = -2$, we have that

$a_4 = 1$, $a_6 = -\frac{1}{3}$, $a_8 = \frac{1}{4 \cdot 3}$, $a_{10} = \frac{-1}{5 \cdot 4 \cdot 3}$

observe that we have the formula

$\boxed{a_{2n} = (-1)^n \cdot \frac{2}{n!}}$ for $n \geq 1$

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Then $$y = \underbrace{\sum_{n=1}^{\infty} (-1)^n \frac{2}{n!} x^{2n}}_{2e^{-x^2}-1} + 1 = \boxed{2e^{-x^2}}$$

4 $y' = ye^x = f(x,y)$, $h = 0.1$

$x_0 = 0$, $y_0 = 1$

Euler method says

$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1}) \cdot h$$

Then

$\underline{n=1}$ $y_1 = y_0 + f(x_0, y_0) \cdot h$

$x_1 = 0.1$ and $y_1 = 1 + f(0,1) \cdot 0.1 = 1.1$

$\underline{n=2}$ $y_2 = 1.1 + f(0.1, 1.1) \cdot 0.1 = 1.2215$

$\underline{n=3}$ $y_3 = 1.2215 + f(0.2, 1.2215) = 1.3706$

$\underline{n=4}$ $y_4 = 1.3706 + f(0.3, 1.3706) = 1.5556$

5) Compute $\int_0^2 e^{\sin(x)}\,dx$ with error $\varepsilon < 0.001$

The error estimate for Simpson's method we use

$$\varepsilon = \frac{(b-a)^5}{180n^4} f^{4)}(c) \quad \text{with} \quad c \in [a, b]$$

Then using the estimate of $f^{4)}(c)$ we have that

$$\varepsilon = \frac{(2-0)^5}{180n^4} \cdot 4 \cdot e^1$$

with $\underline{n=2}$ we have that $\varepsilon = 0.1208$
with $\underline{n=4}$ we have that $\varepsilon = 0.007$
with $\underline{n=6}$ we have that $\varepsilon = 0.0014$
with $\underline{n=8}$ we have that $\varepsilon = 0.0004$

We use Simpsons method with $\underline{n=8}$

then $h = \frac{2-0}{8} = \frac{1}{4}$ and

$$S_8 := \frac{1/4}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + 2f_6 + 4f_7 + f_8\right]$$

$$= \frac{1}{12}\left(e^{\sin 0} + 4e^{\sin 0.25} + 2e^{\sin 0.5} + 4e^{\sin 0.75} + 2e^{\sin 1}\right.$$
$$\left. + 4 \cdot e^{\sin 1.25} + 2e^{\sin 1.5} + 4e^{\sin 1.75} + e^{\sin 2}\right) =$$

$$\boxed{= 4.2366}$$

6 $f(x) = e^{-1/x}$

$f'(x) = e^{-1/x} \frac{1}{x^2}$

$f''(x) = -\frac{2}{x^3} e^{-1/x} + \frac{1}{x^4} e^{-1/x}$

$f^{3)}(x) = \frac{6}{x^4} e^{-1/x} - \frac{2}{x^4} e^{-1/x} + \frac{1}{x^6} e^{-1/x} - \frac{4}{x^5} e^{-1/x}$

$f(1) = e^{-1}$

$f'(1) = e^{-1}$

$f''(1) = -2e^{-1} + e^{-1} = -e^{-1}$

$f'''(1) = 6e^{-1} - 2e^{-1} + e^{-1} - 4e^{-1} = e^{-1}$

$$\boxed{P_3(x) = e^{-1} + e^{-1}(x-1) - \frac{e^{-1}}{2}(x-1)^2 + \frac{e^{-1}}{6}(x-1)^3}$$

$$\int_1^2 f(x)\,dx = \int_1^2 P_3(x) + \frac{f^{4)}(c)}{4!}(x-1)^4\,dx$$

$$\left| \int_1^2 f(x)\,dx - \int_1^3 P_3(x)\,dx \right| \leq \left| \int_1^2 \frac{f^{4)}(c)}{4!}(x-1)^4\,dx \right| <$$

$$< \left| \int_1^2 \frac{e^{-1}}{4!}(x-1)^4\,dx \right| = \frac{e^{-1}}{24} \left| \left[\frac{(x-1)^5}{5}\right]_1^2 \right| = \frac{e^{-1}}{120} =$$

$$\boxed{= 0.003}$$

Then

$$\int_1^2 P_3(x)\,dx = e^{-1}\int_1^2 1+(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}\,dx$$

$$= e^{-1}\left(x+\frac{(x-1)^2}{2}-\frac{(x-1)^3}{6}+\frac{(x-1)^4}{24}\right]_1^2$$

$$= e^{-1}\left(2+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-1\right) = e^{-1}\cdot\frac{33}{24} = \boxed{0.505}$$

b) $\sum_{n=1}^{\infty}\frac{1}{n\cdot 3^n}x^n$

we compute $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)3^{n+1}}}{\frac{1}{n3^n}} =$

$$= \lim_{n\to\infty}\frac{n}{(n+1)\cdot 3} = \boxed{\frac{1}{3}}$$

Then the radius of convergence is $\boxed{r=3}$

We need to check what happens when

$x=\pm 3$

If $x=3 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n3^n}3^n = \sum_{n=1}^{\infty}\frac{1}{n} = \infty$

If $x=-3 \Rightarrow \sum_{n=1}\frac{1}{n3^n}(-3)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} < \infty$

Then $\sum_{n=1}^{\infty} \frac{1}{n3^n} x^n$ converges when $-3 \leq x < 3$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

We compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

Then $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$ converges for every x..

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(7) $f_n(x) = \frac{nx}{1+n^2x^2}$

Given $x \in (0,\infty)$ $\lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$

So $f_n(x) \to 0$ pointwise.

$f_n(x) \not\to 0$ uniformly because one can see that $d(f_n, 0) = 1/2$ $\forall n$.