1. 1) False
2) True
3) True
4) False
5) True
6) True
7) False
8) True
9) True
10) False

2. a) Since $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. So there exists $N > 0$ such that $|a_n| < 1$ $\forall n > N$. Then $|a_n| > a_n^2$ $\forall n > N$. So since $\sum_{n=N}^{\infty} |a_n| < \infty$ so does $\sum_{n=N}^{\infty} a_n^2$ by comparison. $\blacksquare$

b) Since $\lim_{n\to\infty} n^2 a_n = 1$ there exists $N > 0$ and $R > 0$ such that $|n^2 a_n| = \frac{|a_n|}{1/n^2} < R$ $\forall n > N$, but then $|a_n| < R \cdot \frac{1}{n^2}$ $\forall n > N$. Then $\sum_{n=N}^{\infty} |a_n| < R \sum_{n=N}^{\infty} \frac{1}{n^2}$

But since $\sum_{n=N}^{\infty} \frac{1}{n^2}$ converges, so does $\sum_{n=N}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ $\blacksquare$

c) Since $\ln(n) < n \quad \forall n > 1$, so $\frac{1}{n} < \frac{1}{\ln(n)}$, and by comparison $\sum\limits_{n=1} \frac{1}{\ln(n)}$ diverges.

d) First observe $0 < n^2 < e^{n^2} \quad \forall n \geq 0$ and hence $0 < e^{-n^2} < \frac{1}{n^2}$, and by comparison $\sum\limits_{n=1}^{\infty} e^{-n^2}$ converges, and since $|e^{-n^2}| = e^{-n^2}$ we have that it is absolut convergent.

e) We use ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{(n+1)}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} =$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1.$$ Derfor series diverges.

③ $z^5 = -i = 1 \cdot e^{(3\pi/2 + 2\pi k)i} \quad \forall k \in \mathbb{Z}$

$\rightsquigarrow z = e^{\left(3\pi/10 + \frac{2\pi}{5}k\right)i} \quad \forall k \in \mathbb{Z}$

$k=0 \rightsquigarrow \frac{3\pi}{10} \rightsquigarrow z = e^{3\pi/10\, i}$

$k=1 \rightsquigarrow \frac{7\pi}{10} \rightsquigarrow z = e^{7\pi/10\, i}$

$k=2 \rightsquigarrow \frac{11\pi}{10} \rightsquigarrow z = e^{11\pi/10\, i}$

$k=3 \rightsquigarrow \frac{15\pi}{10} \rightsquigarrow z = e^{15\pi/10\, i}$

$k=4 \rightsquigarrow \frac{19\pi}{10} \rightsquigarrow z = e^{19\pi/10\, i}$

④ Take $a \in [0,1]$, the

$\lim_{n \to \infty} a^{2n/n+1} = a^2$

so $f_n(x) \to x^2$ pointwise

Observe that $\{x^{2n/n+1}\}$ is a descending sequence of functions, and since $\{x^{2n/n+1}\}$ converges pointwise in a closed interval $[0,1]$, Dini's theorem says that $\{f_n\}$ must converge uniform.

⑤ 1) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad -1<x<1$

$\downarrow$

$\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \qquad -1<x<1$

$\downarrow$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \arctan x \qquad -1\leq x\leq 1$

$\downarrow$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} (2x)^{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} 2^{2n+1} \cdot x^{2n+1} = \arctan 2x \qquad -\frac{1}{2} \leq x \leq \frac{1}{2}$

2) $f(x) = \ln x \leadsto f'(x) = \frac{1}{x} \leadsto f''(x) = \frac{-1}{x^2}$

$f^{(n)}(x) = \frac{(-1)^n (n-1)!}{x^n} \leadsto f^{(n)}(1) = (-1)^n (n-1)!$

$a_n = \frac{f^{(n)}(1)}{n!} = (-1)^n \frac{1}{n} \qquad n>0$

Then $\ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x-1)^n$

3) $f(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \forall x \in \mathbb{R}$

$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} x^{2n} \quad \forall x \in \mathbb{R}$

$\cos 2x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 4^n x^{4n} \quad \forall x \in \mathbb{R}$

6) $y' = xy^2 = f(x,y)$ & $y(0) = 1$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ $\quad h = 0.1$

$y_0 = 1$

$y_1 = y_0 + 0.1 \cdot f(x_0, y_0) = 1 + 0.1\, f(0,1) = 1$

$y_2 = y_1 + 0.1\, f(x_1, y_1) = 1 + 0.1\, f(0.1, 1) = 1 + 0.1 \cdot 0.1 \cdot 1^2$

$= 1 + 0.01 = 1.01$

$y_3 = y_2 + 0.1\, f(x_2, y_2) = 1.01 + 0.1\, f(0.2, 1.01) =$

$= 1.01 + 0.1 \cdot 0.2 \cdot (1.01)^2 = 1.01 + 0.0204 = 1.0304$

7 $\begin{cases} y''-4y-5y=\sin 2x \\ y(0)=0 \\ y'(0)=0 \end{cases}$

$\lambda^2-4\lambda-5=0 \Rightarrow \lambda=5,-1$

$y_h=Ae^{5x}+Be^{-x}$

$y_p=C\cos 2x+D\sin 2x \rightsquigarrow y_p'=-2C\sin 2x+2D\cos 2x$

$y_p''=-4C\cos 2x-4D\sin 2x$

$-4C\cos 2x-4D\sin 2x+8C\sin 2x-8D\cos 2x-5C\cos 2x-5D\sin 2x$

$=\cos 2x(-4C+8D-5C)+\sin 2x(-4D+8C-5D)=\sin 2x$

$\left.\begin{matrix}-9C+8D=1 \\ -9D+8C=0\end{matrix}\right\} C=\frac{9}{8}D \rightsquigarrow \left(\frac{-9^2}{8}+8\right)D=1$

$\frac{-81+64}{8}D=1$

$\boxed{D=\frac{-8}{17}}$

$\boxed{C=\frac{-9}{17}}$

$\boxed{y_p=\frac{-9}{17}\cos 2x-\frac{8}{17}\sin 2x}$

$y = y_p + y_h$

$y = \frac{-9}{17}\cos 2x - \frac{8}{17}\sin 2x + Ae^{5x} + Be^{-x}$

$y' = \frac{18}{17}\sin 2x - \frac{16}{17}\cos 2x + 5Ae^{5x} - Be^{-x}$

$y(0) = -\frac{9}{17} + A + B = 0 \rightsquigarrow A + B = \frac{9}{17}$

$y'(0) = -\frac{16}{17} + 5A - B = 0 \rightsquigarrow 5A - B = \frac{16}{17}$

$\left.\begin{matrix} \\ \end{matrix}\right\} \rightsquigarrow 6A = \frac{25}{17}$

$\rightsquigarrow A = \frac{25}{102}$

$\rightsquigarrow B = \frac{29}{102}$

$$\rightsquigarrow \boxed{y = -\frac{9}{17}\cos 2x - \frac{8}{17}\sin 2x + \frac{25}{102}e^{5x} + \frac{29}{102}e^{-x}}$$

2) $y = \sum_{n=0}^{\infty} a_n x^n \rightsquigarrow y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$

$y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n$

$$\begin{cases} xy'' - y' + 4x^3y = 0 \\ y(0) = 1 \rightsquigarrow a_0 = 1 \\ y'(0) = 0 \rightsquigarrow a_1 = 0 \end{cases}$$

$$x\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 4x^3\sum_{n=0}^{\infty}a_nx^n = 0$$

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}4a_nx^{n+3} = 0$$

$$\sum_{n=1}^{\infty}n(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=3}^{\infty}4a_{n-3}x^n = 0$$

$$2a_2x + 2\cdot 3a_3x^2 + \sum_{n=3}^{\infty}n(n+1)a_{n+1}x^n - a_1 - 2a_2x - 3a_3x^2$$

$$-\sum_{n=3}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=3}^{\infty}4a_{n-3}x^n = 0$$

$$-a_1 + 3a_3x^2 + \sum_{n=3}^{\infty}\left(n(n+1)a_{n+1} - (n+1)a_{n+1} + 4a_{n-3}\right)x^n = 0$$

$$\begin{bmatrix} a_1 = 0 \\ a_3 = 0 \end{bmatrix}$$

$$n(n+1)a_{n+1} - (n+1)a_{n+1} + 4a_{n-3} = 0$$

$$(n+1)(n-1)a_{n+1} + 4a_{n-3} = 0$$

$$\boxed{a_{n+1} = \frac{-4}{(n+1)(n-1)}a_{n-3}}$$