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$$\textcircled{1} \quad 4x^2 - 5y^2 - 5 = 0$$

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$$4x^2 - 5y^2 = 5$$

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$$\frac{4x^2}{5} - \frac{5y^2}{5} = 1$$

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$$\frac{x^2}{\frac{5}{4}} - y^2 = 1$$

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$$\frac{x^2}{(\frac{\sqrt{5}}{2})^2} - \frac{y^2}{1^2} = 1$$

5°

$$a = \frac{\sqrt{5}}{2} \approx 1.1180$$

$$b = 1$$

$$\begin{aligned} e &= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{\frac{5}{4}}} = \sqrt{1 + \frac{4}{5}} = \sqrt{\frac{9}{5}} = \\ &= \frac{3}{\sqrt{5}} \approx 1,3416 \end{aligned}$$

$$\textcircled{2} \quad a) \quad f_n(x) = \frac{n^2}{1+n^2x^2+n^2} \quad \text{definiert auf } [-1,1]$$

gilt  $x \in [-1,1]$

$$\lim_{n \rightarrow \infty} \frac{n^2}{1+n^2x^2+n^2} = \frac{1}{x^2+1}$$

$$\begin{aligned} d_{x \in [-1,1]}(f_n, \frac{1}{x^2+1}) &= \sup_{x \in [-1,1]} \left| \frac{n^2}{1+n^2x^2+n^2} - \frac{1}{x^2+1} \right| \\ &= \frac{1}{2(1+2n^2)} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

so  $f_n \rightarrow \frac{1}{1+x^2}$  uniform

$$b) \quad f_n(x) = n \sin\left(\frac{x}{n}\right)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(x) &= \lim_{n \rightarrow \infty} n \sin\left(\frac{x}{n}\right) = \\ &= \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{x}{n}\right)}{\frac{1}{n}} \stackrel{\text{Höpital}}{=} \lim_{n \rightarrow \infty} \frac{-\frac{x}{n^2} \cos\frac{x}{n}}{-\frac{1}{n^2}} = \\ &= \lim_{n \rightarrow \infty} x \cdot \cos\left(\frac{x}{n}\right) = x \end{aligned}$$

$$d_{x \in \mathbb{R}}(f_n, x) = \sup_{x \in \mathbb{R}} |x - n \sin\left(\frac{x}{n}\right)| = \infty$$

Se  $f_n$  konvergerer ikke uniformt.

③ a)  $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{\sqrt{n^2+n+1}}$  Divergerer

Vi bruker divergenstest siden

$$\lim_{n \rightarrow \infty} (-1)^n \frac{n+1}{\sqrt{n^2+n+1}} \neq 0.$$

b)  $\sum_{n=1}^{\infty} \frac{(n^2+n)^n}{(3n^2+1)^{2n}}$  Konvergerer

Vi bruker rottest.

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{(n^2+n)^n}{(3n^2+1)^{2n}}} = \lim_{n \rightarrow \infty} \frac{n^2+n}{(3n^2+1)^2} = 0 -$$

c)  $\sum_{n=1}^{\infty} \frac{(n+1)^n}{(2n)!!}$  Konvergerer.

Vi bruker forholdstest.

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{\frac{(n+2)^{n+1}}{(2(n+1))!}}{\frac{(n+1)^n}{(2n)!}} = \lim_{n \rightarrow \infty} \frac{(n+2)^{n+1}}{(n+1)^n (2n+1)(2n+2)} \\
 &= \lim_{n \rightarrow \infty} \left( \frac{n+2}{n+1} \right)^{n+1} \frac{(n+2)(n+1)}{(2n+1)(2n+2)} = \\
 &= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n+1} \right)^{n+1} \frac{(n+2)(n+1)}{(2n+1)(2n+2)} = \frac{e}{4} < 1
 \end{aligned}$$

d)  $\sum_{n=10}^{\infty} \frac{\sqrt{n} \sin(n) + \cos(n^2)}{n^2+n}$  konvergerer

$$\text{Siden } \left| \frac{\sqrt{n} \sin(n) + \cos(n^2)}{n^2+n} \right| \leq \frac{|\sqrt{n} \sin(n)| + |\cos(n^2)|}{|n^2+n|}$$

$$\leq \frac{\sqrt{n} + 1}{n^2+n}$$

$$\text{Så } \sum_{n=10}^{\infty} \left| \frac{\sqrt{n} \sin(n) + \cos(n^2)}{n^2+n} \right| \leq \sum_{n=10}^{\infty} \frac{\sqrt{n} + 1}{n^2+n}$$

↑  
konvergent.

$$④ f(x) = \frac{1}{2} \ln(4+x^2) = \frac{1}{2} \ln\left(\frac{1+\frac{x^2}{4}}{1/4}\right)$$

$$= \frac{1}{2} \ln\left(1 + \frac{x^2}{4}\right) - \frac{1}{2} \ln\left(\frac{1}{4}\right) =$$

$$= \frac{1}{2} \ln\left(1 + \frac{x^2}{4}\right) + \frac{1}{2} \ln 4 =$$

$$= \frac{1}{2} \ln\left(1 + \frac{x^2}{4}\right) + \ln 2$$

Det er nok også finne MacLaurinsrekka til

$$\ln\left(1 + \frac{x^2}{4}\right)$$

$$\frac{1}{1-x} \sim \sum_{n=0}^{\infty} x^n \quad R=1$$

$$\frac{1}{1+x} \sim \sum_{n=0}^{\infty} (-1)^n x^n \quad R=1$$

$$\ln(1+x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} \quad R=1$$

$$\ln\left(1 + \frac{x^2}{4}\right) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}} x^{2n+2} \quad R=2$$

Derfor

$$\frac{1}{2} \ln(4+x^2) = \ln 2 + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}} x^{2n+2}$$

$$(5) \quad e^{i(a+b+c)} = \cos(a+b+c) + i \sin(a+b+c)$$

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$$e^{ia} e^{ib} e^{ic} = (\cos a + i \sin a)(\cos b + i \sin b)(\cos c + i \sin c)$$

$$= (\cos a \cos b - \sin a \sin b + i(\cos a \sin b + \sin a \cos b))(\cos c + i \sin c)$$

$$= \cos a \cos b \cos c - \sin a \sin b \cos c - \cos a \sin b \sin c - \sin a \cos b \sin c \\ + i(\cos a \sin b \cos c + \sin a \cos b \cos c + \cos a \cos b \sin c - \sin a \sin b \sin c)$$

Da

$$\boxed{\cos(a+b+c) = \cos a \cos b \cos c - \sin a \sin b \cos c - \cos a \sin b \sin c - \sin a \cos b \sin c}$$

og

$$\boxed{\sin(a+b+c) = \cos a \sin b \cos c + \sin a \cos b \cos c + \cos a \cos b \sin c - \sin a \sin b \sin c}$$

(6)

$$y'' - 4y' + 4y = e^{2x} \quad \& \quad y(0) = 1 \quad \& \quad y'(0) = 1$$

$$y'' - 4y' + 4y = 0$$

$$\lambda^2 - 4\lambda + 4 = 0 \rightsquigarrow \text{bare 1 røt} \quad \boxed{\lambda = 2}$$

Så  $y_h = (Ax + B)e^{2x}$

Nå vi prøver å finne en løsning av

$$y'' - 4y' + 4y = e^{2x}$$

Vi tipper  $y_p = C \cdot x^2 e^{2x}$

$$y'_p = C \cdot 2x e^{2x} + C \cdot 2 \cdot x^2 e^{2x} = e^{2x} (2Cx + 2Cx^2)$$

$$\begin{aligned} y''_p &= 2C \cdot e^{2x} + 4Cx e^{2x} + 4Cx^2 e^{2x} + 4Cx^2 e^{2x} \\ &= e^{2x} (2C + 8Cx + 4Cx^2) \end{aligned}$$

$$e^{2x} (2C + 8Cx + 4Cx^2) - 4e^{2x} (2Cx + 2Cx^2) + 4Cx^2 e^{2x} =$$

$$= 2C e^{2x} = e^{2x} \Rightarrow \boxed{C = \frac{1}{2}}$$

Så  $y_p = \frac{1}{2} x^2 e^{2x}$

Dø  $y = \frac{1}{2} x^2 e^{2x} + (Ax + B)e^{2x}$

$$y = e^{2x} \left( \frac{1}{2}x^2 + Ax + B \right)$$

$$y(0) = e^0 (C+0+B) = \boxed{B=1}$$

$$y'(x) = 2e^{2x} \left( \frac{1}{2}x^2 + Ax + 1 \right) + e^{2x}(x+A)$$

$$y'(0) = 2e^0 (0+1) + e^0 \cdot A = 2+A = 1$$

$$\Rightarrow \boxed{A=-1}$$

$$\Rightarrow \boxed{y = e^{2x} \left( \frac{1}{2}x^2 - x + 1 \right)}$$

$$\textcircled{7} \quad y'' + xy' + y = 0 \quad \text{der} \quad y = \sum_{n=0}^{\infty} a_n x^n$$

$$y(0)=1 \quad \& \quad y'(0)=0.$$

$$y' = \sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n =$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n =$$

$$a_0 + 2a_2 + \sum_{n=1}^{\infty} ((n+2)(n+1) a_{n+2} + n a_n + a_n) x^n =$$

$$(a_0 + 2a_2) + \sum_{n=1}^{\infty} ((n+2)(n+1) a_{n+2} + (n+1) a_n) x^n = 0$$

Derfor

$$a_0 + 2a_2 = 0$$

og

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$a_{n+2} = -\frac{a_n}{n+2}$$

$$y(0)=1 \Rightarrow \boxed{a_0=1}$$

$$y'(0)=0 \Rightarrow \boxed{a_1=0}$$

$$a_0 + 2a_2 = 0 \Rightarrow 1 + 2a_2 = 0 \Rightarrow \boxed{a_2 = -\frac{1}{2}}$$



$$\boxed{a_3 = \frac{a_1}{3} = 0}$$

$$\boxed{a_5 = \frac{-a_3}{5} = 0} \Rightarrow \boxed{a_{2n+1} = 0}$$

$$a_4 = -\frac{a_2}{4} = \frac{1}{4 \cdot 2} = \frac{1}{2^2(2 \cdot 1)} = \frac{1}{2^2 \cdot 2!}$$

$$a_6 = -\frac{a_4}{G} = -\frac{1}{G} \cdot \frac{1}{2^2 \cdot 2!} = -\frac{1}{2^3 \cdot 3!}$$

$$a_{2n} = \frac{(-1)^n}{2^n \cdot n!}$$

$$\Rightarrow \boxed{y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} x^{2n} = e^{-x^2/2}}$$

(8)  $(x^2 + 1)y' - xy + 1 = 0 \quad \& \quad y(0) = 1$

$$y' = \frac{xy - 1}{x^2 + 1} = f(x, y)$$

$$x_0 = 0 \quad \& \quad y_0 = 1$$

$$x_1 = 0.1 \quad \& \quad \boxed{y_1 = f + 0.1 \cdot f(0,1) = 1 - 0.1 \cdot 1 = \underline{0.9}}$$

$$x_2 = 0.2 \quad \& \quad y_2 = 0.9 + 0.1 \cdot f(0.1, 0.9) \approx \boxed{\underline{0.8099}}$$

$$x_3 = 0.3 \quad \& \quad y_3 = \underline{0.8099} + 0.1 \cdot f(0.2, 0.8099) \\ = \boxed{\underline{0.7293}}$$