Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1102 Grunnkurs i analyse II Vår 2022

Løsningsforslag — Eksamen Vår 2022

1 **a)** Write $z = re^{i\theta}$ where $r \geq 0$ so that

$$z^4 = r^4 e^{i4\theta} = -16 = 16e^{i\pi}.$$

We then deduce that

$$r^4 = 16 \implies r = 2, \qquad 4\theta = \pi + 2\pi n \implies \theta = \frac{\pi}{4} + \frac{\pi n}{2}$$

where n is an integer. This leads to the four solutions

$$z_1 = 2e^{i\pi/4}, \qquad z_2 = 2e^{i3\pi/4}, \qquad z_3 = 2e^{i5\pi/4}, \qquad z_4 = 2e^{7\pi/4}$$

which can also be written as

$$z_1 = \sqrt{2}(1+i), \qquad z_2 = \sqrt{2}(-1+i), \qquad z_3 = \sqrt{2}(-1-i), \qquad z_4 = \sqrt{2}(1-i).$$

b) Since the differential equation is linear and homogenous, we look at its characteristic equation

$$\lambda^4 + 16\lambda = 0$$

which is what we solved in the first part. We can write the solution compactly as $\lambda = \sqrt{2}(\pm 1 \pm i)$ from which we see that the general solution is of the form

$$y(x) = c_1 e^{\sqrt{2}x}\cos(\sqrt{2}x) + c_2 e^{\sqrt{2}x}\sin(\sqrt{2}x) + c_3 e^{-\sqrt{2}x}\cos(\sqrt{2}x) + c_4 e^{-\sqrt{2}x}\sin(\sqrt{2}x).$$

2 We want to choose an A such that $f^{-1}(A)$ captures the subset of $\mathbb{R}$ where f is discontinuous since otherwise $f^{-1}(A)$ would be open. The discontinuity happens at $x = \pi$ and $f(\pi) = \pi$ so we choose $A = (3, 4)$. Then

$$f^{-1}((3,4)) = (3, \pi]$$

which is not open.

3 **a)** Proving that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence means proving that for every $\varepsilon > 0$ we can find a positive integer N such that

$$|x_n - x_m| < \varepsilon$$

whenever $m, n > N$. Since $|b_j - a_j| \to 0$, we can find, given $\varepsilon > 0$, N so large that $(b_j - a_j) < \varepsilon$ for $j \geq N$. Then if $n, m > N$,

$$|x_n - x_m| \leq b_N - a_N < \varepsilon$$

since x_n, x_m both lie in $[a_N, b_N]$.

b) Recall that every Cauchy sequence in $\mathbb{R}$ is convergent, hence there exists a limit point $x = \lim_{j\to\infty} x_j$. We claim that $\{x\} = \cap_{j=1}^{\infty}[a_j, b_j]$. That $x \in \cap_{j=1}^{\infty}[a_j, b_j]$ means that for each $j \in \mathbb{N}$, $x \in [a_j, b_j]$. This holds because all the points x_n for $n \geq j$ lie in $[a_j, b_j]$ which is a closed interval.

Next we claim that x is the only element of $\cap_{j=1}^{\infty}[a_j, b_j]$. Indeed, suppose $x^* \in \cap_{j=1}^{\infty}[a_j, b_j]$. Then for each j,

$$|x - x^*| \leq |b_j - a_j|$$

since $x, x^* \in [a_j, b_j]$. Letting $j \to \infty$ in the above, we see that $|x - x^*| \leq 0$ and so we conclude that $x = x^*$.

4 **a)** Suppose $|x| > 1$ and let $a_n = x^n e^{-\sqrt{n}x}$. We show that $\sum_n a_n$ diverges using the root test as

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |x|e^{-x/\sqrt{n}} = |x|.$$

We conclude that the series diverges when $|x| > 1$. It only remains to check the case $x = -1$ when the series becomes

$$\sum_{n=1}^{\infty}(-1)^n e^{\sqrt{n}}.$$

This series is clearly divergent since its terms do not tend to 0 as $n \to \infty$.

b) We claim that for each $0 < a < 1$, f is continuous on $[-a, 1]$ This will yield the desired conclusion since the union of all these intervals is $(-1, 1]$.

To show the continuity we use Weierstrass' M-test. Fix $0 < a < 1$ and note that when $x \in [-a, 1]$,

$$|x^n e^{-\sqrt{n}x}| \leq a^n e^{\sqrt{n}a} + e^{-\sqrt{n}a} =: M_n$$

by the helpful inequality in the question. Weierstrass' M-test will yield that f is continuous on $[-a, 1]$ if we can show that

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} a^n e^{\sqrt{n}a} + \sum_{n=1}^{\infty} e^{-\sqrt{n}a} < \infty.$$

Here the first sum is finite by the root test calculation we did in a). We see that second sum is finite for example by observing that $e^{-\sqrt{n}a} \leq 1/n^2$ for sufficiently large n, so that convergence follows by comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges.

5 **a)** We present two possible solutions, one using an induction argument and another using complex exponentials.

For the induction argument, the base case $N = 1$ is immediate. Now for the induction step, suppose that

$$\sum_{n=1}^{N-1} \sin(an) = \frac{\sin((N-1)a/2)\sin(Na/2)}{\sin(a/2)}.$$

It then holds that

$$\begin{aligned}\sum_{n=1}^{N}\sin(an) &= \sum_{n=1}^{N-1}\sin(an)+\sin(aN)\\ &= \frac{\sin((N-1)a/2)\sin(Na/2)}{\sin(a/2)}+\sin(aN)\\ &= \frac{\sin((N-1)a/2)\sin(Na/2)+\sin(aN)\sin(a/2)}{\sin(a/2)}.\end{aligned}$$

Hence the desired conclusion will follow if we can show that the numerators agree, i.e.,

$$\sin(Na/2)\sin((N+1)a/2) = \sin((N-1)a/2)\sin(Na/2)+\sin(aN)\sin(a/2).$$

Using the double angle formula $\sin(2x) = 2\sin(x)\cos(x)$ on the last $\sin(aN)$ yields

$$\begin{aligned}&\sin((N-1)a/2)\sin(Na/2)+\sin(aN)\sin(a/2)\\ &\qquad = \sin(Na/2)\Big(\sin((N-1)a/2)+2\cos(Na/2)\sin(a/2)\Big)\end{aligned}$$

and so it only remains to show that

$$\sin((N+1)a/2) = \sin((N-1)a/2)+2\cos(Na/2)\sin(a/2).$$

We rewrite the left hand side using the sine addition formula and the right hand side using the sine subtraction formula, yielding

$$\begin{aligned}LHS &= \sin(Na/2+a/2) = \sin(Na/2)\cos(a/2)+\cos(Na/2)\sin(a/2),\\ RHS &= \sin(Na/2-a/2)+2\cos(Na/2)\sin(a/2)\\ &= \sin(Na/2)\cos(a/2)-\cos(Na/2)\sin(a/2)+2\cos(Na/2)\sin(a/2)\\ &= \sin(Na/2)\cos(a/2)+\cos(Na/2)\sin(a/2) = LHS\end{aligned}$$

which finishes the proof.

For the complex exponential argument, recall that $\sin(x) = \mathrm{Im}(e^{ix})$. Thus

$$\sum_{n=1}^{N}\sin(an) = \mathrm{Im}\left(\sum_{n=1}^{N}e^{ian}\right) = \mathrm{Im}\left(\frac{e^{ia}(1-e^{iaN})}{1-e^{ia}}\right)$$

where we used that the sum is geometric. From here we use the relation

$$e^{ix}-e^{-ix} = 2i\sin(x)$$

to rewrite

$$\begin{aligned}\frac{e^{ia}(1-e^{iaN})}{1-e^{ia}} &= \frac{e^{ia/2}(1-e^{iaN})}{e^{-ia/2}-e^{ia/2}} = \frac{e^{ia/2}e^{iaN/2}(e^{-iaN/2}-e^{iaN/2})}{-2i\sin(a/2)}\\ &= \frac{e^{ia/2}e^{iaN/2}(-2i\sin(aN/2))}{-2i\sin(aN/2)}\\ &= \frac{\sin(aN/2)}{\sin(a/2)}e^{ia(N+1)/2}\\ \implies \mathrm{Im}\left(\sum_{n=1}^{N}e^{ian}\right) &= \frac{\sin(aN/2)}{\sin(a/2)}\mathrm{Im}\left(e^{ia(N+1)/2}\right)\\ &= \frac{\sin(aN/2)\sin(a(N+1)/2)}{\sin(a/2)}\end{aligned}$$

which is what we wished to show.

b) We may use directly Thm. 3.31 in Krantz, although it was not taught in class. Alternatively, following the hint, we use partial summation on integral form. We write

$$a_n = \sin(an), \quad b_n = \frac{1}{\sqrt{n}},$$
$$A(N) = \sum_{n=1}^{N} a_n = \frac{\sin(Na/2)\sin((N+1)a/2)}{\sin(a/2)}, \quad A(x) = A(j) \text{ for } j \le x < j+1,$$
$$f(x) = \frac{1}{\sqrt{x}} \implies f'(x) = \frac{-1}{2x^{3/2}}$$

and note that for all N, $|A(N)| \le \frac{1}{|\sin(a/2)|} < \infty$. We now write out the partial summation as

$$\begin{aligned}\sum_{n=1}^{N} a_n f(n) &= A(N)f(N) - \int_0^N A(x)f'(x)dx \\ &= \frac{\sin(aN/2)\sin(a(N+1)/2)}{\sin(a/2)}\frac{-1}{2N^{3/2}} + \frac{1}{2}\int_0^N A(x)\frac{1}{x^{3/2}}dx.\end{aligned}$$

By the bound on $A(N)$ established above, the first term is bounded by a constant divided by $N^{3/2}$ which clearly goes to zero as $N \to \infty$. The integral term is bounded by $\frac{1}{2|\sin(a/2)|}\int_1^\infty \frac{1}{x^{3/2}}dx = \frac{1}{|\sin(a/2)|} < \infty$ (where we used that $A(x) = 0$ for $x < 1$) and hence the sum must be convergent.

6 We first show that the statement in the hint is true. Since the degree of P is less than or equal to 10, the degree of P' is less than or equal to 9. Hence P' can have no more than 9 zeros. Consequently, at least one of the 10 intervals

$$[0,1), [1,2), \ldots, [9,10)$$

must be free of zeros of P' which is the statement in the hint.

Let $[k, k+1)$ be such an interval. We wish to show that $\max_{x\in[0,10]} |\sin(\pi x) - P(x)| \ge 1/2$ which certainly will follow if we prove that the difference is $\ge 1/2$ on $[k, k+1)$. Suppose towards a contradiction that $|\sin(\pi x) - P(x)| < 1/2$ on $[k, k+1)$. There are 4 different cases possible depending on the signs of $\sin(\pi x)$ and $P'(x)$ on $[k, k+1)$. We show that all four cases lead to a contradiction by using that

$$\sin(\pi k) = \sin(\pi(k+1)) = 0 \implies \frac{-1}{2} < P(k),\ P(k+1) < \frac{1}{2}$$
$$\sin(\pi(k+1/2)) = \pm 1 \implies |P(k+1/2)| > \frac{1}{2}.$$

Case 1: $P'(x) > 0$, $\sin(\pi x) > 0$:

$$\frac{1}{2} < P(k+1/2) < P(k+1) < \frac{1}{2}.$$

Case 2: $P'(x) < 0$, $\sin(\pi x) > 0$:

$$\frac{1}{2} > P(k) > P(k+1/2) > \frac{1}{2}.$$

Case 3: $P'(x) > 0$, $\sin(\pi x) < 0$:

$$\frac{-1}{2} < P(k) < P(k+1/2) < \frac{-1}{2}.$$

Case 4: $P'(x) < 0$, $\sin(\pi x) < 0$:

$$\frac{-1}{2} > P(k+1/2) > P(k+1) > \frac{-1}{2}.$$

Since all cases lead to contradictions, we deduce that our assumption $|\sin(\pi x) - P(x)| < 1/2$ was false which yields the desired conclusion.