Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1102
Grunnkurs i analyse II
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Løsningsforslag — Øving 6

1 In the lectures we saw that the infinite union of open sets are open. For this exercise, we prove the corresponding result for closed sets and give counterexamples.

a) We prove that the complement $\left(\bigcap_n E_n\right)^c$ is open by showing that

$$\left(\bigcap_n E_n\right)^c = \bigcup_n E_n^c$$

which in turn is open by virtue of being an infinite union of open sets. The above equality is one of De Morgan's laws which is a result from logic.

For the $\subset$ inclusion, note that a point x is in $\left(\bigcap_n E_n\right)^c$ if and only if it is not in the intersection of all E_n. This is equivalent to there existing a E_{n_0} such that $x \notin E_{n_0}$ meaning that $x \in (E_{n0})^c \subset \bigcup_n E_n^c$.

For the reverse inclusion, note that if $x \in \bigcup_n E_n^c$, then $x \in E_{n_0}^c \implies x \notin E_{n_0}$. But the set E_{n_0} is a superset of $\bigcap E_n$ so x is also not in the smaller set $\bigcap E_n$ meaning that $x \in \left(\bigcap E_n\right)^c$.

b) Let $E_n = \left[\frac{1}{2n}, 1 - \frac{1}{2n}\right]$, then the union is $(0, 1)$ because any $0 < \varepsilon < 1/2$ is contained in E_n for $n > \frac{1}{2\varepsilon}$. Meanwhile the endpoints are certainly not in the union because $0 < \frac{1}{2n}$ for all n.

c) Let $U_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$, then the intersection is $\{0\}$ because any $\varepsilon > 0$ is eventually outside U_n for $n > \frac{1}{\varepsilon}$ while $0 \in U_n$ for all n.

2 **a)** The set is $\mathbb{Z}$ which is closed and unbounded, hence not compact.

b) The real line is both open and closed but not bounded and so not compact.

c) The rationals are neither open nor closed and hence their complement is neither open nor closed.

d) This set is also not closed not open.

3 Assume to the contrary that

$$d(K, L) = \inf_{k \in K, l \in L} d(k, l) = 0 \implies \text{there exists } (x_n)_n \subset K, (y_n)_n \subset L \text{ such that } d(x_n, y_n) \to 0.$$

Then since K is compact, $(x_n)_n$ has a convergent subsequence $x_{n_m} \to x \in K$. Then

$$d(x, y_{n_m}) \leq d(x, x_{n_m}) + d(x_{n_m}, y_{n_m}) \to 0 \text{ as } m \to \infty$$

and so $x \in K$ is a limit point of L. By the closedness of L, this implies that K, L are not disjoint and so we have reached a contradiction.

$\boxed{4}$ **a)**

$$(0, \infty)$$

b)

$$(-1, 0) \quad \text{and} \quad (0, 1)$$

c)

$$\{0\}$$

d)

$$\bigcup_n \left\{\frac{1}{n}\right\} \cup \{0\}$$

$\boxed{5}$ **a)** We prove that the complement of $\overline{A}$ is open. Let x be a point not in $\overline{A}$. We claim that there exists a $\varepsilon > 0$ such that

$$B_\varepsilon(x) \cap A = \emptyset.$$

Indeed, if such a ε did not exist, then for each $\varepsilon > 0$ we could find a a_ε such that $d(x, a_\varepsilon) < \varepsilon$. This would induce a sequence $(a_n)_n$ in A that converges to x which contradicts $x \notin \overline{A}$.
If we can show that there exists a ball centered at x of radius $\varepsilon/2$ that does not intersect $\overline{A}$, we are done as this implies that the complement of $\overline{A}$ is open. To see that this holds, assume towards a contradiction that

$$B_{\varepsilon/2}(x) \cap \overline{A} \neq \emptyset.$$

Call this element y and let $(a_n)_n$ denote the sequence of points in A that converges to y. Since this sequence converges, we can find a $N > 0$ such that $d(y, a_N) < \varepsilon/2$. We then have that

$$d(a_N, x) \leq d(a_N, y) + d(y, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

but this means that $B_\varepsilon(x) \cap A \neq \emptyset$, a contradiction!
The conclusion is that $B_{\varepsilon/2}(x) \cap \overline{A} = \emptyset$ and this implies that the complement of $\overline{A}$ indeed is open and we are done.

b) The closure of the rationals is all of $\mathbb{R}$. Indeed, any real number has a decimal expansion so let r_n be the rational number whose decimal expansion agrees with that of x for the first n digits. The sequence $(r_n)_n$ will converge to x as for any $\varepsilon > 0$, we can find a n so large that $|x - r_n| < \varepsilon$.