

- 1 In the lectures we saw that the infinite union of open sets are open. For this exercise, we prove the corresponding result for closed sets and give counterexamples.
 - a) We prove that the complement $\left(\bigcap_{n} E_{n}\right)^{c}$ is open by showing that

$$\left(\bigcap_{n} E_{n}\right)^{c} = \bigcup_{n} E_{n}^{c}$$

which in turn is open by virtue of being an infinite union of open sets. The above equality is one of De Morgan's laws which is a result from logic.

For the \subset inclusion, note that a point x is in $\left(\bigcap_n E_n\right)^c$ if and only if it is not in the intersection of all E_n . This is equivalent to there existing a E_{n_0} such that $x \notin E_{n_0}$ meaning that $x \in (E_{n_0})^c \subset \bigcup_n E_n^c$.

For the reverse inclusion, note that if $x \in \bigcup_n E_n^c$, then $x \in E_{n_0}^c \implies x \notin E_{n_0}$. But the set E_{n_0} is a superset of $\bigcap E_n$ so x is also not in the smaller set $\bigcap E_n$ meaning that $x \in (\bigcap E_n)^c$.

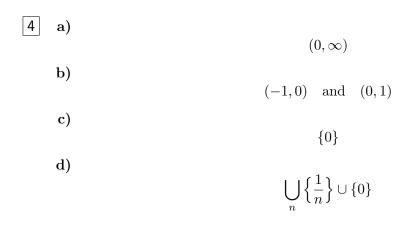
- **b)** Let $E_n = \left[\frac{1}{2n}, 1 \frac{1}{2n}\right]$, then the union is (0, 1) because any $0 < \varepsilon < 1/2$ is contained in E_n for $n > \frac{1}{2\varepsilon}$. Meanwhile the endpoints are certainly not in the union because $0 < \frac{1}{2n}$ for all n.
- c) Let $U_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$, then the intersection is $\{0\}$ because any $\varepsilon > 0$ is eventually outside U_n for $n > \frac{1}{\varepsilon}$ while $0 \in U_n$ for all n.
- **a)** The set is \mathbb{Z} which is closed and unbounded, hence not compact.
 - b) The real line is both open and closed but not bounded and so not compact.
 - c) The rationals are neither open nor closed and hence their complement is neither open nor closed.
 - d) This set is also not closed not open.
- 3 Assume to the contrary that

$$d(K,L) = \inf_{k \in K, l \in L} d(k,l) = 0 \implies \text{ there exists } (x_n)_n \subset K, (y_n)_n \subset L \text{ such that } d(x_n, y_n) \to 0.$$

Then since K is compact, $(x_n)_n$ has a convergent subsequence $x_{n_m} \to x \in K$. Then

 $d(x, y_{n_m}) \le d(x, x_{n_m}) + d(x_{n_m}, y_{n_m}) \to 0 \text{ as } m \to \infty$

and so $x \in K$ is a limit point of L. By the closedness of L, this implies that K, L are not disjoint and so we have reached a contradiction.



5

a) We prove that the complement of \overline{A} is open. Let x be a point not in \overline{A} . We claim that there exists a $\varepsilon > 0$ such that

$$B_{\varepsilon}(x) \cap A = \emptyset.$$

Indeed, if such a ε did not exist, then for each $\varepsilon > 0$ we could find a a_{ε} such that $d(x, a_{\varepsilon}) < \varepsilon$. This would induce a sequence $(a_n)_n$ in A that converges to x which contradicts $x \notin \overline{A}$.

If we can show that there exists a ball centered at x of radius $\varepsilon/2$ that does not intersect \overline{A} , we are done as this implies that the complement of \overline{A} is open. To see that this holds, assume towards a contradiction that

$$B_{\varepsilon/2}(x) \cap \overline{A} \neq \emptyset.$$

Call this element y and let $(a_n)_n$ denote the sequence of points in A that converges to y. Since this sequence converges, we can find a N > 0 such that $d(y, a_N) < \varepsilon/2$. We then have that

$$d(a_N, x) \le d(a_N, y) + d(y, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

but this means that $B_{\varepsilon}(x) \cap A \neq \emptyset$, a contradiction! The conclusion is that $B_{\varepsilon/2}(x) \cap \overline{A} = \emptyset$ and this implies that the complement of \overline{A} indeed is open and we are done.

b) The closure of the rationals is all of \mathbb{R} . Indeed, any real number has a decimal expansion so let r_n be the rational number whose decimal expansion agrees with that of x for the first n digits. The sequence $(r_n)_n$ will converge to x as for any $\varepsilon > 0$, we can find a n so large that $|x - r_n| < \varepsilon$.