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Department of Mathematical Sciences

Løsningsforslag — Øving 12

1 **a)** Assume that we had an x for which

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

By taking the derivative on both sides, we find that

$$x'(t) = f(t, x(t))$$

and verifying that $x(t_0) = x_0$ is trivial.

b) We need to verify that $T(y) \in C(J)$ and that $\sup_{t\in J} |x_0 - T(y)(t)| \le c\beta$. That $T(y)$ is continuous follows from that f is continuous on R. We further verify that

$$\sup_{t\in J} |x_0 - T(y)(t)| = \sup_{t\in J} \left| \int_{t_0}^{t} f(s, y(s))\, ds \right| \le \int_{t_0}^{t_0+\beta} c\, ds = c\beta.$$

c) For each $t \in J$ (which is where y_1, y_2 are defined), we have that

$$\begin{aligned} |T(y_1)(t) - T(y_2)(t)| &= \left| \int_{t_0}^{t} [f(s, y_1(s)) - f(s, y_2(s))]\, ds \right| \\ &\le \int_{t_0}^{t} |f(s, y_1(s)) - f(s, y_2(s))|\, ds \\ &\le \int_{t_0}^{t_0+\beta} k \sup_{t\in J} |y_1(t) - y_2(t)| = k\beta d_\infty(y_1, y_2). \end{aligned}$$

d) To apply Banach's fixed point theorem, we only need that T is contractive which is guaranteed by $k\beta < 1 \iff \beta < \frac{1}{k}$.

e) Applying Banach's fixed point theorem, we get that there exists an $x \in X$ such that $T(x) = x$. By the result in a), the desired conclusion follows.

2 On the square $|t| < 1, |x| < 1$, the function $f(t, x) = -tx$ is clearly continuous and bounded. The Lipschitz constant for f in this square is 1 which is finite and thus the conditions of Picard-Lindelöf are satisfied.

For the Picard iteration, we start with $x_0(t) = 1$ to match the initial condition and find

$$\begin{aligned} x_1(t) &= 1 - \int_0^t s \cdot 1\, ds = 1 - \frac{t^2}{2}, \\ x_2(t) &= 1 - \int_0^t s \cdot (1 - s^2/2)\, ds = 1 - \frac{t^2}{2} + \frac{t^4}{2\cdot 4}, \\ x_3(t) &= 1 - \int_0^t s \cdot (1 - s^2/2 + s^4/(2\cdot 4))\, ds = 1 - \frac{t^2}{2} + \frac{t^4}{2\cdot 4} - \frac{t^6}{2\cdot 4\cdot 6}. \end{aligned}$$

The pattern we're supposed to notice is that for each iteration, a new term gets added which is of a predictable format. Would we have continued in this fashion, we would have gotten a series of the form

$$1 - \frac{t^2}{2} + \frac{t^4}{2\cdot 4} - \frac{t^6}{2\cdot 4\cdot 6} + \frac{t^8}{2\cdot 4\cdot 6\cdot 8} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n! 2^n} = \sum_{n=0}^{\infty} \frac{(\frac{-x^2}{2})^n}{n!} = e^{-x^2/2}.$$

3 Using the provided formula, we compute

$$\begin{aligned} y(0.1) \approx y_1 &= y_0 + \frac{h}{2}\big[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\big] \\ &= 1 + \frac{0.1}{2}\big[0 + 1 + 0.1 + 1 + 0.1 \cdot (0+1)\big] = 1.11, \\ y(0.2) \approx y_2 &= y_1 + \frac{h}{2}\big[f(x_1, y_1) + f(x_2, y_1 + hf(x_1, y_1))\big] \\ &= 1.11 + \frac{0.1}{2}\big[0.1 + 1.11 + 0.2 + 1.11 + 0.1 \cdot (0.1 + 1.11)\big] = 1.24205. \end{aligned}$$

4 **a)** We know that any linear combination of the two solutions is also a solutions, we therefore make the ansatz

$$y(x) = A\sin(2x) + B\cos(2x) \implies y'(x) = 2A\cos(2x) - 2B\sin(2x)$$

which yields

$$B = 2, A = 1/2 \implies y(x) = \frac{1}{2}\sin(2x) + 2\cos(2x).$$

b) Again we make the ansatz

$$y(x) = Ae^{-\frac{1}{2}x} + Bxe^{-\frac{1}{2}x} \implies y'(x) = \frac{-1}{2}e^{-x/2}[A + B(x-2)].$$

From $y'(2) = 2$ we get that $A = -4e$ and from $y(2) = 0$ we get $A + 2B = 0$ which yields

$$y(x) = -4e^{-x/2+1} + 2xe^{-x/2+1}.$$

5 **a)** We compute the derivatives of y as

$$\begin{aligned} y(x) &= Ax^2 + Bx + C \\ \implies y'(x) &= 2Ax + B \\ \implies y''(x) &= 2A. \end{aligned}$$

Plugging this into the differential equation yields

$$\begin{aligned} y'' + 3y' - y &= 2A + 6Ax + 3B - Ax^2 - Bx - C \\ &= (-A)x^2 + (6A - B)x + (2A + 3B - C). \end{aligned}$$

Putting the above to be equal to $4x$ force the values $A = 0, B = -4, C = -12$ and so we get

$$y(x) = -4x - 12$$

as the solution.

b) We compute the derivatives of y as

$$\begin{aligned} y(x) &= A\sin(2x) + B\cos(2x) \\ \implies y'(x) &= 2A\cos(2x) - 2B\sin(2x) \\ \implies y''(x) &= -4A\sin(2x) - 4B\cos(2x). \end{aligned}$$

Plugging this into the differential equation yields

$$\begin{aligned} y'' + 2y' - 2y &= -4A\sin(2x) - 4B\cos(2x) + 4A\cos(2x) - 4B\sin(2x) - 2A\sin(2x) - 2B\cos(2x) \\ &= (-4A - 4B - 2A)\sin(2x) + (-4B + 4A - 2B)\cos(2x). \end{aligned}$$

Putting the above to be equal to $\sin(2x)$ force the values $A = \frac{-3}{26}, B = \frac{-1}{13}$ and so we get

$$y(x) = \frac{-3}{26}\sin(2x) - \frac{1}{13}\cos(2x)$$

as the solution.

c) We compute the derivatives of y as

$$\begin{aligned} y(x) &= (A + Bx)e^x \\ \implies y'(x) &= (A + B + Bx)e^x \\ \implies y''(x) &= (A + 2B + Bx)e^x. \end{aligned}$$

Plugging this into the differential equation yields

$$y'' + 8y' - 6y = (A + 2B + Bx + 8A + 8B + 8Bx - 6A - 6Bx)e^x$$

Putting the above to be equal to xe^x force the values $A = \frac{-10}{9}, B = \frac{1}{3}$ and so we get

$$y(x) = \frac{-10e^x}{9} + \frac{xe^x}{3}.$$

as the solution.