MA1102
Grunnkurs i analyse II
Vår 2023

Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

Løsningsforslag — Øving 11

1 **a)**

$$\sin(ix) = \frac{e^{iix} - e^{-iix}}{2i} = \frac{-i}{2}(e^{-x} - e^{x}) = \frac{i}{2}(e^{x} - e^{-x}) = i\sinh(x).$$

b)

$$\begin{aligned}\cos(a+b) &= \mathrm{Re}(e^{i(a+b)}) = \mathrm{Re}(e^{ia}e^{ib}) \\ &= \mathrm{Re}((\cos(a) + i\sin(a))(\cos(b) + i\sin(b))) \\ &= \cos(a)\cos(b) - \sin(a)\sin(b).\end{aligned}$$

c)

$$\begin{aligned}\sin(2x) &= \mathrm{Im}(e^{i2x}) = \mathrm{Im}((e^{ix})^2) \\ &= \mathrm{Im}((\cos(x) + i\sin(x)^2)) \\ &= \cos(x)\sin(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x).\end{aligned}$$

2 Recall that Newton's method can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

a) We have $f'(x) = 3x^2 + 2$ and so we get

$$\begin{aligned}x_0 &= 0, \\ x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-1}{2} = \frac{1}{2}, \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{1}{2} - \frac{\frac{1}{8} + 1 - 1}{\frac{3}{4} + 2} = \frac{10}{22}.\end{aligned}$$

b) We approximate the minimum value by searching for a zero of $f'(x) = 2x - 2 - \sin(2x)$. To do so we also need to compute $f''(x) = 2 - 2\cos(2x)$ which yields

$$x_1 = 1 - \frac{2 \cdot 1 - 2 - \sin(2 \cdot 1)}{2 - \cos(2 \cdot 1)} \approx 1.3763.$$

We thus approximate the minimum value as

$$f(x_1) \approx f(1.3763) \approx 0.1782.$$

3 **a)** This is a subsequence of $(x_0^n)_n$ which converges to zero as $n \to \infty$ which can be seen by e.g. noting that the sum $\sum_n x_0^n$ is convergent and hence the terms must approach zero.

b) This time we use the Banach fixed point theorem. By the mean value theorem we have that

$$|T(x) - T(y)| \leq C|x - y|$$

if C is a bound for the derivative of $\frac{1}{1+x^2}$. Using standard methods, we can verify that $C < 1$ and so the desired conclusion follows by Banach's fixed point theorem.

c) Here we cannot use the same approach directly since the derivative of cosine can take the value 1. To get around this, note that the first iterate $\cos(x_0) \in [-1, 1]$ and $|\frac{d}{dx}\cos(x)| < 1$ for $x \in [0, 1]$ and so we can apply the fixed point theorem on $[-1, 1]$.

4 **a)** Our desired operator is

$$T(f)(x) = \lambda \int_a^b k(x, y)f(y)\,dy + g(x).$$

b) We estimate

$$\begin{aligned} |T(f)(x) - T(g)(x)| &= \left|\lambda \int_a^b k(x-y)[f(y) - g(y)]\,dy\right| \\ &\leq |\lambda| \int_a^b |k(x, y)||f(y) - g(y)|\,dy \\ &\leq |\lambda|\|kd(f, g) \int_a^b dy = |\lambda|\|kd_\infty(f, g)|b - a| \end{aligned}$$

where we used that $|k(x, y)| \leq \|k\|_\infty < \infty$ and the triangle inequality for integrals twice.

c) To apply the Banach fixed point theorem and get existance of a solution, we need the constant $|\lambda|\|k\|_\infty|b - a|$ to be less than 1.

d) In this case $|b - a| = 2$, $|\lambda| = \frac{1}{2\pi}$, $\|k\|_\infty = 1$ and so the above condition is satisfied.