Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1102 Grunnkurs i analyse II Vår 2023

Løsningsforslag — Øving 1

1 Let b, b' be two least upper bounds for $X \subset \mathbb{R}$. Then since b' is an upper bound and b is a *least* upper bound,

$$b \leq b'.$$

Similarly, since b is an upper bound and b' is a *least* upper bound,

$$b' \leq b.$$

Putting this together we get

$$b \leq b' \leq b \implies b = b'$$

and the least upper bound is unique.

Let l, l' be two greatest lower bounds for $X \subset \mathbb{R}$. Then since l' is a lower bound and l is a greatest lower bound,

$$l' \leq l.$$

Similarly, since l is a lower bound and l' is a greatest lower bound,

$$l \leq l'.$$

Putting this together we get

$$l' \leq l \leq l' \implies l = l'$$

and the greatest lower bound is unique.

2 **a)** For any $a \in A$ and $b \in B$, $a \leq \sup A$ and $b \leq \sup B$ and so for each element $c \in A + B$, $c \leq \sup A + \sup B$ which implies that

$$\sup(A + B) \leq \sup A + \sup B.$$

Next suppose towards a contradiction that $\sup(A+B) < \sup A + \sup B$ and let $\varepsilon = \sup A + \sup B - \sup(A + B) > 0$. By the definitions of $\sup A$ and $\sup B$ we can find $a \in A$ and $b \in B$ such that

$$a > \sup A - \frac{\varepsilon}{2},$$
$$b > \sup B - \frac{\varepsilon}{2}$$

Therefore, using the definition of ε,

$$a + b > \sup A + \sup B - \varepsilon > \sup(A + B)$$

which contradicts the definition of $\sup(A + B)$. We therefore conclude that we must have the equality $\sup(A + B) = \sup A + \sup B$.

b) For any $a \in A$ and $b \in B$, $a \leq \sup A$ and $b \leq \sup B$ and so for each element $c \in A \cdot B$, $c \leq (\sup A) \cdot (\sup B)$ which implies that

$$\sup(A \cdot B) \leq (\sup A) \cdot (\sup B).$$

Next suppose towards a contradiction that $\sup(A \cdot B) < (\sup A) \cdot (\sup B)$ and let $\varepsilon = (\sup A) \cdot (\sup B) - \sup(A \cdot B)$.
By the definitions of $\sup A$ and $\sup B$ we can find $a \in A$ and $b \in B$ such that

$$\begin{aligned} a &> \sup A - \frac{\varepsilon}{2 \sup B}, \\ b &> \sup B - \frac{\varepsilon}{2 \sup A}. \end{aligned}$$

Therefore, using the definition of ε,

$$\begin{aligned} a \cdot b &> \left(\sup A - \frac{\varepsilon}{2 \sup B}\right) \cdot \left(\sup B - \frac{\varepsilon}{2 \sup A}\right) \\ &= (\sup A) \cdot (\sup B) - \varepsilon + \frac{\varepsilon^2}{4 (\sup A) \cdot (\sup B)} \\ &> (\sup A) \cdot (\sup B) - \varepsilon \\ &= \sup(A \cdot B) \end{aligned}$$

which contradicts the definition of $\sup(A \cdot B)$. We therefore conclude that we must have the equality $\sup(A \cdot B) = (\sup A) \cdot (\sup B)$.

3 We first note that the point $z = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$ is in the second quadrant. By reflecting along the imaginary axis and computing the argument of the corresponding point in the first quadrant, we find that

$$\arg(z) = \pi - \arg(\frac{1}{2} + \frac{\sqrt{3}}{2} i) = \pi - \arctan(\frac{\sqrt{3}}{1}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Krantz 1.2: 3 We wish to find the all cube roots of $1 + i$. We can write

$$z^3 = 1 + i = r e^{i\theta} = \sqrt{2} e^{i\pi/4}$$

in polar form. Then solutions $z = s e^{i\psi}$ must satisfy $s^3 = r$ and $3\psi = 2\pi \cdot n + \theta$. This yields

$$z_1 = 2^{1/6} e^{i \frac{\pi}{12}}, \qquad z_2 = 2^{1/6} e^{i \frac{3\pi}{4}}, \qquad z_3 = 2^{1/6} e^{i \frac{17\pi}{4}}.$$

Krantz 1.2: 7 Let p be a polynomial with real coefficients, i.e.,

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad a_m \in \mathbb{R}, \quad m = 0, 1, \ldots, n.$$

Then if $p(z) = 0$,

$$p(\bar{z}) = a_0 + a_1 \bar{z} + \cdots + \bar{z}^n = \overline{a_0 + a_1 z + \cdots + a_n z^n} = \overline{p(z)} = \bar{0} = 0$$

since $(\bar{z})^n = \overline{(z^n)}$.

Krantz 1.2: 11 We want a picture of

$$S = \{z \in \mathbb{C} : |z-1| + |z+1| = 2\}.$$

If we write $z = x + iy$, we see that if $y \neq 0$,

$$|z-1| + |z+1| = \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} > |x-1| + |x+1| \geq 2$$

where we in the last step used the triangle inequality as

$$2 = |2 - x + x| = |1 - x + (1 + x)| \leq |x-1| + |x+1|.$$

We therefore conclude that $y = 0$ for all z in S. Now if $|x| > 1$, then

$$2 < |2x| = |x + 1 + x - 1| \leq |x+1| + |x-1| = 2$$

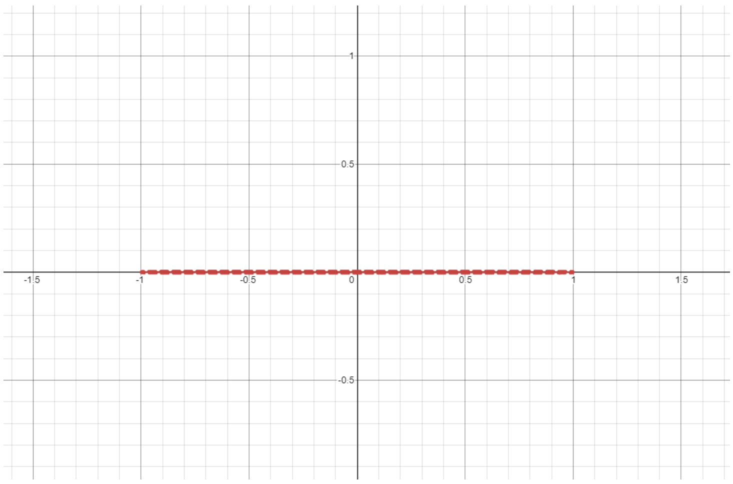
a contradiction! Meanwhile if $0 \leq |x| \leq 1$, then

$$|x-1| = 1 - x \implies |x+1| + |x-1| = x + 1 + (1-x) = 2$$

and so

$$S = \{x + iy \in \mathbb{C} : |x| \leq 1, y = 0\}$$

which we draw as



Krantz 1.2: 14 We wish to find the complex numbers z such that

$$z^2 = -1 - i = \sqrt{2}e^{i5\pi/4}.$$

The solutions $z = se^{i\psi}$ must satisfy $s^2 = \sqrt{2}$ and $2\psi = 5\pi/4 + 2\pi \cdot n$. This yields

$$z_1 = 2^{1/4}e^{i\frac{5\pi}{8}}, \qquad z_2 = 2^{1/4}e^{i\frac{13\pi}{8}}.$$

Krantz 1.2: 18 We want to draw a picture of

$$T = \Big\{z \in \mathbb{C} : |z + \bar{z}| - |z - \bar{z}| = 2\Big\}.$$

If we write $z = x + iy$, we see that

$$z + \bar{z} = x + iy + (x - iy) = 2x, \qquad z - \bar{z} = x + iy - (x - iy) = 2iy.$$

Hence,

$$T = \{x + iy \in \mathbb{C} : 2|x| - 2|y| = 2\} = \{x + iy \in \mathbb{C} : |x| - |y| = 1\}$$

which looks like