Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1102 Grunnkurs i analyse II Vår 2023

Øving 9

1 Define the function f by

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(27^n x)}{3^n}.$$

a) Use the Weierstrass M-test to show that f is continuous for all $x \in \mathbb{R}$.

b) Show that f is **uniformly** continuous on all of $\mathbb{R}$.

Remark: This function is actually not differentiable for any $x \in \mathbb{R}$, imagine that! For a proof (outline) see Øving 7 from last year.

2 Show that

$$f(x) = \sum_{n=0}^{\infty} 3^n \sin\left(\frac{x}{27^n}\right)$$

is a continuous function on all of $\mathbb{R}$.
Hint: $|\sin(x)| \leq |x|$.

3 Show that the function

$$f(x) = \sum_{n=1}^{\infty} e^{-nx}$$

defines a continuous function on $(0, \infty)$.

4 Let $(f_n)_n$ be defined by

$$f_n(x) = \frac{\cos(nx)}{n}.$$

a) Show that $(f_n)_n$ converges uniformly to a function f.

b) Show that $(f_n')_n$ does **not** converge to f'.

5 In the exercises on power series (Øving 5), we showed that

$$x \in D := \left\{ y : |y| < \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} \right\} \implies f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{converges.}$$

a) Show that the above convergence is uniform on any compact subset of D using the Weierstrass M-test.

b) Compute the derivative and primitive of f and show that they also converge uniformly in the same interval.

6 **a)** Find the power series $\ln(1+x) = \sum_{n=0}^{\infty} a_n x^n$ which converges for $|x| < 1$ by showing that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges uniformly and integrating.

b) Express

$$\frac{1}{(1-x)^2}$$

as a power series which converges for $|x| < 1$.
Hint: Derivative of $\frac{1}{1-x}$.