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Department of Mathematical Sciences

Øving 7

1 **a)** Let $f(x) = \cos(x)$, compute $f^{-1}((0,1))$. Is this set open?

b) Let f be defined as

$$f(x) = \begin{cases} \sin(x) & x < 0, \\ e^{-x} + 1 & x \geq 0. \end{cases}$$

Prove that this function is discontinuous by finding an open set U such that $f^{-1}(U)$ is not open.

c) For $f(x) = e^{-x^2}$, compute $f^{-1}([1,2])$.

2 The function $f : \mathbb{R} \setminus \{0\} \to [0,1], x \mapsto \sin(\frac{1}{x})$ is clearly continuous for $x \neq 0$ as it is the composition of two continuous functions. Prove that not f cannot be extended to a continuous function on all of $\mathbb{R}$ by showing that the limit

$$\lim_{x \to 0} f(x)$$

does not exist.

3 **a)** Give an example of a continuous function f and an open set U so that $f(U)$ is not open.

b) Give an example of a discontinuous function f and a closed set E so that $f^{-1}(E)$ is open.

4 If f is continuous on $[0,1]$ and if $f(x)$ is positive for each rational x, then does it follow that f is positive for all x?

5 This exercise deals with fixed points of functions, i.e., points x such that $f(x) = x$.

a) Prove that any continuous function $f : [0,1] \to [0,1]$ has a fixed point.
Hint: Intermediate value theorem.

b) Is the same result true for $f : (0,1) \to (0,1)$? Give a proof or counterexample.

c) Give an example of a discontinuous function $f : [0,1] \to [0,1]$ that has no fixed point. A drawing will suffice.