Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

Øving 5

1 We have previously used but never showed that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

In this exercise, we aim to prove this rigorously. Convergence of the sum was showed in the exercises 2 weeks ago.

To start off, we will assume that $x \geq 0$ for **a)**, **b)**, **c)** and **d)**.

a) In the lectures, we saw that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Extend this argument to show that

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

b) Use the binomial theorem to show that

$$\left(1 + \frac{x}{n}\right)^n \leq \sum_{k=0}^{n} \frac{x^k}{k!}$$

for all $n \in \mathbb{N}$.

c) Modify the above argument to show that for $m \leq n$,

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \geq \sum_{k=0}^{m} \frac{x^k}{k!}$$

Hint: Cut off the binomial sum at the m:th term and take the limit.

d) Combine the three above results to conclude that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Instead of repeating the proof with considerations for the $x < 0$ case, we will manually **verify** that

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

which is a fairly obvious guess in light of the formula for $x \geq 0$. Our main tool will be Theorem 3.49 from Krantz which says that if $\sum_n a_n$ and $\sum_n b_n$ are two absolutely convergent series, then

$$\left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n, \qquad c_n = \sum_{j=0}^{n} a_j b_{n-j}$$

e) Verify that

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

using the above theorem. With this we can conclude that the formula holds for all $x \in \mathbb{R}$.
Hint: Binomial theorem

2 Recall that we have defined complex exponentials as

$$e^{ix} = \cos(x) + i\sin(x)$$

earlier in this course. While we have not shown that the formula from Exercise 1 is valid for complex numbers, assume that it is and examine the consequences by extracting a sum similar to the form $\sum_n a_n x^n$ for both $\cos(x)$ and $\sin(x)$.
Hint: Plug in $x = iy$ and see what happens.

3 Sums of form

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

are commonly called **power series** and notable examples include

$$\sum_{n=0}^{\infty} 1 \cdot (x-0)^n = \frac{1}{1-x}, \qquad \sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n = e^x$$

as well as $\cos x$ and $\sin x$ (see Exercise 1). Note that all power series do not always converge, such as that for $\frac{1}{1-x}$ above and that the center c can be non-zero. We will learn more about these later in the course but we are already in a position to do some computations with them.

a) For which $x \in \mathbb{R}$ does the power series for $\frac{1}{1-x}$ converge? Use a convergence test.

b) For which $x \in \mathbb{R}$ does the power series for e^x converge? Use a convergence test.

4 Knowing the power series for some functions allows us to write up the power series for their close relatives using a change of variables. We do this in the exercises below.

a) Write down the power series for the **Gaussian** $g(x) = e^{-x^2}$. For which x does it converge?

b) In statistics we often use a parametrized form of the Gaussian written as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

and refer to σ as the standard deviation and μ as the mean. Write this function as a power series.

c) Another so-called **probability distribution** used in statistics is the **gamma distribution** which has two parameters, k, θ, called shape and scale. The gamma distribution can be written as

$$f(x) = \frac{1}{\Gamma(k)\theta^{k-1}} x^k e^{-\frac{x}{\theta}}$$

where $\Gamma(k)$ is a constant depending on k. Write this function as a power series assuming that k is a positive integer and $\theta > 0$.

5 In this exercise we will do similar computations with the geometric series which we earlier remarked does not converge everywhere. In the subexercises below, write down a power series for the given function and indicate for which $x \in \mathbb{R}$ the sum converges.

a)

$$\frac{1}{1-x}$$

b)

$$\frac{1}{1+x}$$

c)

$$\frac{1}{2+2x}$$

d)

$$\frac{1}{2+x}$$

e)

$$\frac{x^5}{2+x}$$

f)

$$\frac{x^5}{2+x^2}$$

6 Using the quotient test (forholdstest), formulate a condition involving x and the sequence $(a_n)_{n=0}^{\infty}$ which guarantees that the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

converges.

Note that this should generalize the results in exercise 2.