Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

Øving 4

1 We have seen in the lectures that the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is convergent. Let S denote value which it converges to and S_N the N:th partial sum. We wish to estimate the difference between the partial sums and S, i.e.

$$\left| \sum_{n=1}^{N} \frac{(-1)^n}{n} - S \right| = |S_N - S|.$$

Provide an explicit upper bound for this quantity and give an integer N_0 such that

$$|S_N - S| \leq \frac{1}{1000}$$

for $N > N_0$.

2 Determine if the following series converge or not.

a)

$$\sum_{n=1}^{\infty} \frac{n - \cos n}{n^2 + 2n}$$

b)

$$\sum_{n=1}^{\infty} n^2 e^{-\sqrt{n}}$$

c)

$$\sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}}$$

3 For which $a \in \mathbb{R}$ is the following sum convergent

$$\sum_{n=1}^{\infty} a^n \frac{n^2}{2^n}.$$

Exercises 3, 7 from Krantz 3.3:

Exercises

1. If $1/2 > b_j > 0$ for every j and if $\sum_{j=1}^{\infty} b_j$ converges then prove that $\sum_{j=1}^{\infty} \frac{b_j}{1-b_j}$ converges.

2. Follow these steps to give another proof of the Alternating Series Test: **a)** Prove that the odd partial sums form an increasing sequence; **b)** Prove that the even partial sums form a decreasing sequence; **c)** Prove that every even partial sum majorizes all subsequent odd partial sums; **d)** Use a pinching principle.

3. What can you say about the convergence or divergence of

$$\sum_{j=1}^{\infty} \frac{(2j+3)^{1/2} - (2j)^{1/2}}{j^{3/4}} ?$$

4. For which exponents k and ℓ does the series

$$\sum_{j=2}^{\infty} \frac{1}{j^k |\log j|^\ell}$$

converge?

5. Let p be a polynomial with integer coefficients and degree at least 1. Let $b_1 \geq b_2 \geq \cdots \geq 0$ and assume that $b_j \to 0$. Prove that if $(-1)^{p(j)}$ is not always positive and not always negative then in fact it will alternate in sign so that $\sum_{j=1}^{\infty} (-1)^{p(j)} \cdot b_j$ will converge.

6. Explain in words how summation by parts is analogous to integration by parts.

7. If $\gamma_j > 0$ and $\sum_{j=1}^{\infty} \gamma_j$ converges then prove that

$$\sum_{j=1}^{\infty} (\gamma_j)^{1/2} \cdot \frac{1}{j^\alpha}$$

converges for any $\alpha > 1/2$. Give an example to show that the assertion is false if $\alpha = 1/2$.

* **8.** Assume that $\sum_{j=1}^{\infty} b_j$ is a convergent series of positive real numbers. Let $s_j = \sum_{\ell=1}^{j} b_\ell$. Discuss convergence or divergence for the series $\sum_{j=1}^{\infty} s_j \cdot b_j$. Discuss convergence or divergence for the series $\sum_{j=1}^{\infty} \frac{b_j}{1+s_j}$.

* **9.** If $b_j > 0$ for every j and if $\sum_{j=1}^{\infty} b_j$ diverges then define $s_j = \sum_{\ell=1}^{j} b_\ell$. Discuss convergence or divergence for the series $\sum_{j=1}^{\infty} \frac{b_j}{s_j}$.

* **10.** Let $\sum_{j=1}^{\infty} b_j$ be a conditionally convergent series of real numbers. Let β be a real number. Prove that there is a rearrangement of the series that converges to β. (**Hint:** First observe that the positive terms of the given series must form a divergent series. Also, the negative terms form a divergent series. Now build the rearrangement by choosing finitely many positive terms whose sum "just exceeds" β. Then add on enough negative terms so that the sum is "just less than" β. Repeat this oscillatory procedure.)