Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

Øving 3

1 **a)** Prove that for each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

b) Prove that the sum

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is convergent.

2 If $a_j > 0, b_j > 0, \sum_j a_j$ converges, and $\sum_j b_j$ converges, then what can you say about $\sum_j a_j b_j$?

3 Let p be a polynomial with no constant term. If $b_j > 0$ for every j and if $\sum_j b_j$ converges then prove that the series $\sum_j p(b_j)$ converges.

4 TRUE or FALSE: If a_j and b_j are positive and $\sum_j a_j$ and $\sum_j b_j$ both converge, then $\sum_j a_j b_j$ converges.

Exercises 4, 10 from Krantz 3.3:
Hint for exercise 4: Cauchy criteria

Exercises

1. If $1/2 > b_j > 0$ for every j and if $\sum_{j=1}^{\infty} b_j$ converges then prove that $\sum_{j=1}^{\infty} \frac{b_j}{1-b_j}$ converges.

2. Follow these steps to give another proof of the Alternating Series Test: **a)** Prove that the odd partial sums form an increasing sequence; **b)** Prove that the even partial sums form a decreasing sequence; **c)** Prove that every even partial sum majorizes all subsequent odd partial sums; **d)** Use a pinching principle.

3. What can you say about the convergence or divergence of

$$\sum_{j=1}^{\infty} \frac{(2j+3)^{1/2} - (2j)^{1/2}}{j^{3/4}} ?$$

4. For which exponents k and ℓ does the series

$$\sum_{j=2}^{\infty} \frac{1}{j^k |\log j|^\ell}$$

converge?

5. Let p be a polynomial with integer coefficients and degree at least 1. Let $b_1 \geq b_2 \geq \cdots \geq 0$ and assume that $b_j \to 0$. Prove that if $(-1)^{p(j)}$ is not always positive and not always negative then in fact it will alternate in sign so that $\sum_{j=1}^{\infty} (-1)^{p(j)} \cdot b_j$ will converge.

6. Explain in words how summation by parts is analogous to integration by parts.

7. If $\gamma_j > 0$ and $\sum_{j=1}^{\infty} \gamma_j$ converges then prove that

$$\sum_{j=1}^{\infty} (\gamma_j)^{1/2} \cdot \frac{1}{j^\alpha}$$

converges for any $\alpha > 1/2$. Give an example to show that the assertion is false if $\alpha = 1/2$.

* **8.** Assume that $\sum_{j=1}^{\infty} b_j$ is a convergent series of positive real numbers. Let $s_j = \sum_{\ell=1}^{j} b_\ell$. Discuss convergence or divergence for the series $\sum_{j=1}^{\infty} s_j \cdot b_j$. Discuss convergence or divergence for the series $\sum_{j=1}^{\infty} \frac{b_j}{1+s_j}$.

* **9.** If $b_j > 0$ for every j and if $\sum_{j=1}^{\infty} b_j$ diverges then define $s_j = \sum_{\ell=1}^{j} b_\ell$. Discuss convergence or divergence for the series $\sum_{j=1}^{\infty} \frac{b_j}{s_j}$.

* **10.** Let $\sum_{j=1}^{\infty} b_j$ be a conditionally convergent series of real numbers. Let β be a real number. Prove that there is a rearrangement of the series that converges to β. (**Hint:** First observe that the positive terms of the given series must form a divergent series. Also, the negative terms form a divergent series. Now build the rearrangement by choosing finitely many positive terms whose sum "just exceeds" β. Then add on enough negative terms so that the sum is "just less than" β. Repeat this oscillatory procedure.)

Exercises 2, 5 from Krantz 3.4:

Exercises

1. Use induction to prove the formulas provided in the text for the sum of the first N perfect squares, the first N perfect cubes, and the first N perfect fourth powers.

2. A real number s is called *algebraic* if it satisfies a polynomial equation of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m = 0$$

with the coefficients a_j being integers and $a_m \neq 0$. Prove that, if we replace the word "integers" in this definition with "rational numbers," then the set of algebraic numbers remains the same. Prove that $n^{p/q}$ is algebraic for any positive integers p, q, n.

A number which is not algebraic is called *transcendental.*

3. Discuss convergence of $\sum_j 1/[\ln j]^k$ for k a positive integer.

4. Discuss convergence of $\sum_j 1/p(j)$ for p a polynomial.

5. Discuss convergence of $\sum_j \exp(p(j))$ for p a polynomial.