Norges teknisk–naturvitenskapelige universitet Department of Mathematical Sciences	MA1102 Grunnkurs i analyse II Vår 2023 **Øving 2**

1 Describe geometrically (or make a sketch of) the set of points z in the complex plane satisfying

$$\pi \leq \arg(z) \leq \frac{7\pi}{4}.$$

2 Use de Moivre's Theorem to find a trigonometric identity for $\cos 3\theta$ in terms of $\cos\theta$ and one for $\sin 3\theta$ in terms of $\sin\theta$.

3 Describe the solutions, if any, of the equations

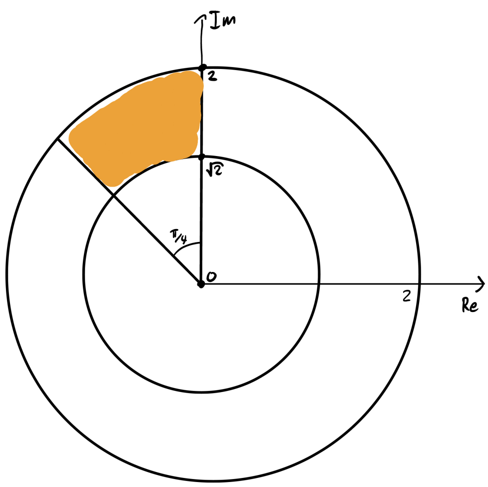
a) $\bar{z} = \frac{2}{z}$.

b) $\bar{z} = -\frac{2}{z}$.

4 Find all complex solutions of the equation

$$z^4 + 1 - i\sqrt{3} = 0.$$

5 Specify the subset of the complex plane colored orange in the following figure:



Give your answer as a set $A = \{z \in \mathbb{C} : ...\}$. Note that the picture is not to scale.

Exercises 2, 12 from Krantz 2.1:

Exercises

1. Suppose a sequence $\{a_j\}$ has the property that, for every natural number N, there is a j_N such that $a_{j_N} = a_{j_{N+1}} = \cdots = a_{j_{N+N}}$. In other words, the sequence has arbitrarily long repetitive strings. Does it follow that the sequence converges?

2. Let α be an irrational real number and let a_j be a sequence of rational numbers converging to α. Suppose that each a_j is a fraction expressed in lowest terms: $a_j = \alpha_j/\beta_j$. Prove that the β_j are unbounded.

3. Let $\{a_j\}$ be a sequence of rational numbers all of which have denominator a power of 2. What are the possible limits of such a sequence?

4. Redo Exercise **3** with the additional hypothesis that all of the denominators are less than or equal to 2^{10}.

5. Use the integral of $1/(1+t^2)$, together with Riemann sums (ideas which you know from calculus, and which we shall treat rigorously later in the book), to develop a scheme for calculating the digits of π.

6. Prove Corollary 2.18.

7. Prove Proposition 2.20.

8. Prove parts **(2)** and **(4)** of Proposition 2.6.

9. Give an example of a decreasing sequence that converges to π.

10. Prove the following result, which we have used without comment in the text: Let S be a set of real numbers which is bounded above and let $t = \sup S$. For any $\epsilon > 0$ there is an element $s \in S$ such that $t - \epsilon < s \le t$. (**Remark:** Notice that this result makes good intuitive sense: the elements of S should become arbitrarily close to the supremum t, otherwise there would be enough room to decrease the value of t and make the supremum even smaller.) Formulate and prove a similar result for the infimum.

11. Let $\{a_j\}$ be a sequence of real or complex numbers. Suppose that every subsequence has itself a subsequence which converges to a given number α. Prove that the full sequence converges to α.

* **12.** Let $\{a_j\}$ be a sequence of complex numbers. Suppose that, for every pair of integers $N > M > 0$, it holds that $|a_M - a_{M+1}| + |a_{M+1} - a_{M+2}| + \cdots + |a_{N-1} - a_N| \le 1$. Prove that $\{a_j\}$ converges.

13. Let $a_1, a_2 > 0$ and for $j \ge 3$ define $a_j = a_{j-1} + a_{j-2}$. Show that this sequence cannot converge to a finite limit.

Exercises 2, 4 from Krantz 2.2:

1. Use the Bolzano–Weierstrass theorem to show that every decreasing sequence that is bounded below converges.

[1]Some books say "converging to infinity," but this terminology can be confusing.

2. Give an example of a sequence of rational numbers with the property that, for any real number α, or for $\alpha = +\infty$ or $\alpha = -\infty$, there is a subsequence approaching α.

3. Prove that if $\{a_j\}$ has a subsequence diverging to $\pm\infty$ then $\{a_j\}$ cannot converge.

4. Let $x_1 = 2$. For $j \geq 1$, set

$$x_{j+1} = x_j - \frac{x_j^2 - 2}{2x_j}.$$

Show that the sequence $\{x_j\}$ is decreasing and bounded below. What is its limit?

5. The sequence

$$a_j = (1 + 1/2 + 1/3 + \cdots + 1/j) - \log j$$

is a famous example. It is known to converge, but nobody knows whether the limit is rational or irrational. Draw a picture which shows that the sequence converges.

6. Provide the details of the proof of Proposition 2.31.

* **7.** Provide the details of the assertion that the sequence $\{\cos j\}$ is dense in the interval $[-1, 1]$.

* **8.** Let n be a positive integer. Consider n, $n + 1$, $\ldots$ modulo π. This means that you subtract from each number the greatest multiple of π that does not exceed it. Prove that this collection of numbers is dense in $[0, \pi]$. That is, the numbers get arbitrarily close to any element of this interval.

* **9.** Let $S = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$. Give an example of a sequence $\{a_j\}$ with the property that, for each $s \in S$, there is a subsequence converging to s, but no subsequence converges to any limit not in S.

* **10.** Give another proof of the Bolzano–Weierstrass theorem as follows. If $\{a_j\}$ is a bounded sequence let $b_j = \inf\{a_j, a_{j+1}, \ldots\}$. Then each b_j is finite, $b_1 \leq b_2 \leq \ldots$, and $\{b_j\}$ is bounded above. Now use Proposition 2.16.

* **11.** Prove that the sequence

$$a_N = \sum_{m=1}^{N} \frac{\sin m}{m}$$

converges.

* **12.** Prove that the sequence

$$a_N = \sum_{m=1}^{N} \frac{\sin^2 m}{m}$$

diverges.