1 In this exercise we will prove the Picard-Lindelöf theorem which gives conditions under which the ODE

$$x'(t) = f(t,x), \qquad x(t_0) = x_0$$

has a solution in a neighborhood of t_0.

Specifically, assume that f is continuous on

$$R = \{(t,x) : |t - t_0| < a,\, |x - x_0| < b\}$$

and bounded by $c \in \mathbb{R}$. Moreover, we assume that f is *Lipschitz* in its second argument which means that there exists a positive constant k such that

$$|f(t,x) - f(t,y)| \leq k|x - y|$$

for all (t,x) and (t,y) in R.

a) Show that in order to show that the ODE has a solution, it suffices to show that there exists a function $x \in C^1[t_0 - a, t_0 + a]$ such that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

Hint: Differentiate; $x \in C^1$ means that x is continuously differentiable.

b) With the interval $J = [t_0 - \beta, t_0 + \beta]$ we can define the closed space

$$X = \left\{ y \in C(J) \,:\, y(t_0) = x_0,\, \sup_{t \in J} |x_0 - y(t)| \leq c\beta \right\}$$

(we assume β is so small that $c\beta < b$). Show that the operator T defined by

$$T(y)(t) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds, \qquad t \in J$$

maps X to X, i.e., $T(y) \in X$ for $y \in X$.

c) Show that for $y_1, y_2 \in X$,

$$d_\infty(T(y_1), T(y_2)) \leq k\beta d_\infty(y_1, y_2).$$

d) Find a condition on β so that we can use Banach's fixed point theorem to conclude that there exists a unique $x \in C(J)$ such that

$$T(x) = x.$$

e) Combine the above results to conclude that

$$x'(t) = f(t, x), \qquad x(t_0) = x_0$$

has a unique solution on $C^1(J)$.

2 Verify that the ODE

$$x'(t) = -tx(t), \qquad x(0) = 1$$

satisfies the conditions of the Picard-Lindelöf theorem and compute the next 3 Picard iterations (x_1, x_2, x_3). Start with $x_0(t) = 1$ to match the initial condition.

Can you recognize which Taylor series this approaches? If not that is okay!
Hint: $2 \cdot 4 \cdot 6 \cdots 2n = 2^n \cdot n!$.

3 Let $y(x)$ be the solution of the ODE

$$y' = x + y, \qquad y(0) = 1.$$

Use the improved Euler method with step size $h = 0.1$ to approximate the values $y(0.1)$ and $y(0.2)$.

Reminder: Recall that the improved Euler method can be written as

$$y_{n+1} = y_n + \frac{h}{2}\big[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\big].$$

4 **a)** We are given that $y_1(x) = \sin(2x)$ and $y_2(x) = \cos(2x)$ are solutions to the homogenous differential equation

$$y'' = -4y.$$

Find the unique solution that satisfies the initial conditions $y(0) = 2, y'(\pi) = 1$.

b) We are given that $y_1(x) = e^{-\frac{1}{2}x}$ and $y_2(x) = xe^{-\frac{1}{2}x}$ are solutions to the homogenous differential equation

$$4y'' + 4y' + y = 0$$

Find the unique solution that satisfies the initial conditions $y(2) = 0, y'(2) = 2$.

5 When solving inhomogenous differential equations of the form

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x),$$

the most efficient approach may be to guess a solution. In the exercises below, you are given good guesses which share similarities with the function $f(x)$ - this is in general a good approach.

a) For which values of the constants A, B, C is the function $y(x) = Ax^2 + Bx^2 + C$ a solution to the differential equation.

$$y'' + 3y' - y = 4x.$$

b) For which values of the constants A, B is the function $y(x) = A\sin(2x) + B\cos(2x)$ a solution to the differential equation.

$$y'' + 2y' - 2y = \sin(2x).$$

c) For which values of the constants A, B is the function $y(x) = Ae^x + Bxe^x$ a solution to the differential equation.

$$y'' + 8y' - 6y = xe^x.$$