Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1102
Grunnkurs i analyse II
Vår 2023

Øving 11

1 Use Euler's formula for sine and cosine to rewrite the following quantities into nicer forms.

a) $\sin(ix)$.

b) $\cos(a+b)$.

c) $\sin(2x)$.

2 **a)** Compute two iterations of Newton's method to approximate a zero of the following function

$$f(x) = x^3 + 2x - 1$$

on the interval $[0, 1]$ with the starting point $x_0 = 0$.

b) Use Newton's method to approximate the minimum value of the function

$$f(x) = (x-1)^2 + \cos(x)^2$$

starting at the initial guess $x_0 = 1$. You only need to compute one iteration.

3 **a)** Prove that the sequence $(x_0, T(x_0), T(T(x_0)), T(T(T(x_0))), \dots)$ converges for $T(x) = x^2$ and $|x_0| < 1$.

b) Prove that the sequence $(x_0, T(x_0), T(T(x_0)), T(T(T(x_0))), \dots)$ converges for $T(x) = \frac{1}{1+x^2}$ and any $x_0 \in \mathbb{R}$.

c) Prove that the sequence $(x_0, T(x_0), T(T(x_0)), T(T(T(x_0))), \dots)$ converges for $T(x) = \cos(x)$ and any $x_0 \in \mathbb{R}$.

4 In this exercise we will show that **integral equations** of the form

$$f(x) = \lambda \int_a^b k(x,y) f(y)\, dy + g(x) \tag{1}$$

have solutions under certain conditions. Throughout, we will assume that the functions g and k are continuous on $[a, b] \times [a, b]$.

Our tool will be a version the Banach fixed point theorem which is more general than the one covered in the lectures. If X is a complete space with metric (distance function) d and T is a contractive mapping, i.e.,

$$d(T(x,y)) \leq d(x,y) \qquad \text{for all } x, y \in X,$$

then there is a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.

In our case, the space X will be $C[a, b]$ and the distance function is the same one we used for convergence of function sequences

$$d_\infty(f, g) = \|f - g\|_\infty = \max_{x \in [a,b]} |f(x) - g(x)|.$$

a) Find an operator $T : C[a, b] \to C[a, b]$ such that (1) reads as $T(f) = f$.

b) Show that

$$d_\infty(T(f), T(g)) \le \|k\|_\infty |\lambda|(b - a) d_\infty(f, g).$$

Hint: Use that k is a continuous function on a compact domain and the triangle inequality for integrals: $\left|\int kf\, dx\right| \le \|k\|_\infty \int |f|\, dx$.

c) What condition do we need to place on k, λ, a and b to get the existence of a solution?

d) Prove that a solution exists to

$$f(x) = \frac{1}{2\pi} \int_{-1}^{1} \frac{f(y)}{1 + x^2 + y^2}\, dy + x^2$$

(but don't try to find it.)