Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1102
Grunnkurs i analyse II
Vår 2023

Øving 1

1 Prove that the least upper bound and greatest lower bound for a set of real numbers is unique.

2 Prove the following properties of the supremum.

a) If A and B are subsets of the reals and $A+B=\{a+b : a\in A, b\in B\}$, then

$$\sup A + \sup B = \sup(A+B).$$

b) If A, B are bounded subsets of the positive real line and $A\cdot B=\{a\cdot b : a\in A, b\in B\}$, then

$$(\sup A)\cdot(\sup B) = \sup(A\cdot B).$$

Hint: A version of $(x-\frac{\varepsilon}{2y})(y-\frac{\varepsilon}{2x}) > xy-\varepsilon$ *can be useful.*

3 Find the argument of $z=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$.

Exercises 3, 7, 11, 14, 18 from the following exercises in Krantz: (see next page)

3. Find all cube roots of the complex number $1 + i$.

4. Taking the commutative, associative, and distributive laws of addition and multiplication for the real number system for granted, establish these laws for the complex numbers.

5. Consider the function $\phi : \mathbb{R} \to \mathbb{C}$ given by $\phi(x) = x + i \cdot 0$. Prove that ϕ respects addition and multiplication in the sense that $\phi(x + x') = \phi(x) + \phi(x')$ and $\phi(x \cdot x') = \phi(x) \cdot \phi(x')$.

6. Prove that the field of complex numbers cannot be made into an *ordered* field. (**Hint:** Since $i \neq 0$ then either $i > 0$ or $i < 0$. Both lead to a contradiction.)

7. Prove that the complex roots of a polynomial with real coefficients occur in complex conjugate pairs.

8. Calculate the square roots of i.

9. Prove that the set of all complex numbers is uncountable.

10. Prove that any nonzero complex number z has kth roots $r_1, r_2, \ldots, r_k$. That is, prove that there are k of them.

11. In the complex plane, draw a picture of

$$S = \{z \in \mathbb{C} : |z - 1| + |z + 1| = 2\}.$$

12. Refer to Exercise 9. Show that the kth roots of z all lie on a circle centered at the origin, and that they are equally spaced.

13. Find all the cube roots of $1 + i$.

14. Find all the square roots of $-1 - i$.

15. Prove that the set of all complex numbers with rational real part is uncountable.

16. Prove that the set of all complex numbers with both real and imaginary parts rational is countable.

17. Prove that the set $\{z \in \mathbb{C} : |z| = 1\}$ is uncountable.

* **18.** In the complex plane, draw a picture of

$$T = \{z \in \mathbb{C} : |z + \overline{z}| - |z - \overline{z}| = 2\}.$$