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1. **True** ($A = [0, 1]$)
2. **True** ($B \subset \mathbb{Q}$ and $\mathbb{Q}$ is countable)
3. **True** (Infinite intersection of closed is compact, intersection of bounded is bounded)
4. **False** (e.g. $K_n = [n, n + 1]$)
5. **True** (Cauchy sequences are convergent, sequence is in $[0, 1]$ which is closed)
6. **False** (e.g. $x_n = \frac{\sqrt{2}}{n}$)
7. **True** (look at complements)
8. **False** (e.g. $f(x) = \frac{1}{2}$)
9. **True** (image of compact is compact)
10. **True** (continuous functions preserve limits)

2

1. Consider the partial sums

$$s_N = \sum_{n=1}^{N}(a_n - a_{n-1}) = a_1 - a_0 + a_2 - a_1 + a_3 - a_2 + \cdots + a_n - a_{n-1} = a_n - a_0.$$

   Since $(a_n)_n$ is Cauchy and hence convergent, so is $(s_n)_n$ and hence the sum is convergent.

2. Since $(a_n)_n$ is bounded, there exists a constant C such that $|a_n| \leq C$ for all n. Therefore we have that

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{2^n}\right| \leq \sum_{n=1}^{\infty}\frac{C}{2^n} = C < \infty.$$

3. The sum $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent. Indeed, we do not have absolute convergence as $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent but the conditions of the alternating series test are satisfied as the series is alternating, terms go to zero as $n \to \infty$ and $\frac{1}{\sqrt{n}}$ is decreasing in n.

4. The sum $\sum_{n=1}^{\infty}\frac{n^2}{n!}$ is convergent as can be seen by e.g. the ratio test. Indeed,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 n!}{n^2 n!(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{n^2(n+1)}\right| = 0.$$

5. Note that $\cos(n\frac{2\pi}{2}) = \cos(n\pi) = (-1)^n$ and hence the sum is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Again this sum is conditionally convergent as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges but the conditions of the alternating series test are fulfilled.

3 Following the hint, we set $w = z^3$ and solve $w^2 + w + 1 = 0 \implies w = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$. We now solve the two equations

$$z^3 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i = e^{i2\pi/3},$$
$$z^3 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i = e^{i4\pi/3}$$

which we have converted to polar form. If we write $z = e^{i\theta}$, this yields the two equations

$$3\theta = \frac{2\pi}{3} + 2\pi n \implies \theta = \frac{2\pi}{9} + \frac{6\pi n}{9}$$
$$3\theta = \frac{4\pi}{3} + 2\pi n \implies \theta = \frac{4\pi}{9} + \frac{6\pi n}{9}$$

for $n = 0, 1, 2$. The values are then

$$z_1 = e^{i2\pi/9},$$
$$z_2 = e^{i8\pi/9},$$
$$z_3 = e^{i14\pi/9},$$
$$z_4 = e^{i4\pi/9},$$
$$z_5 = e^{i10\pi/9},$$
$$z_6 = e^{i16\pi/9.}$$

4 For $x = 0$, $f_n(x) = \frac{n}{n+1}$ which converges to 1 as $n \to \infty$. For $x \in (0, 1]$, the exponential dominates and $f_n(x) \to 0$ as $n \to \infty$. Hence f_n converges pointwise to the function

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

As each f_n is continuous but f is not, the convergence cannot be uniform.

5 1. We know that

$$\ln(1 + y) = -\sum_{n=1}^{\infty} \frac{(-1)^n y^n}{n} \qquad \text{for } |y| < 1$$

as can be verified by integrating the power series representation of $\frac{1}{1-x}$ from the formula collection. Plugging in $y = 2x^2$ yields

$$\ln(2x^2+1) = -\sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^{2n}}{n} \qquad \text{for } |2x^2| < 1 \iff |x| < \frac{1}{\sqrt{2}}.$$

2. We know that

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \qquad \text{for all } y$$

from the formula collection. Plugging in $y = -x/3$ yields

$$xe^{-x/3} = x\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{3^{n-1}(n-1)!} \qquad \text{for all } x.$$

3. Instead of solving a system of equations to determine $a_0, \dots, a_4$, we compute the power series of f centered at $x = 1$. This requires us to compute

$$\begin{aligned} f(x) &= x^4 + x^3 + x^2 + x + 1 && \implies f(1) = 5, \\ f'(x) &= 4x^3 + 3x^2 + 2x + 1 && \implies f'(1) = 10, \\ f''(x) &= 12x^2 + 6x + 2 && \implies f''(1) = 20, \\ f'''(x) &= 24x + 6 && \implies f'''(1) = 30, \\ f''''(x) &= 24 && \implies f''''(1) = 24. \end{aligned}$$

We can now write the power series as

$$f(x) = \sum_{n=0}^{4} \frac{f^{(n)}(1)}{n!}(x-1)^n = 5 + 10(x-1) + 10(x-1)^2 + 5(x-1)^3 + (x-1)^4.$$

The convergence area is all of $\mathbb{R}$ as our power series are finite.

6 To use Newton's method we need to compute the derivative

$$f'(x) = 2 + \sin(x).$$

Iterations are now compute using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Starting with $x_0 = 0$, we get $x_1 = 0.5$, $x_2 \approx 0.4506$, $x_3 \approx 0.4502$.

7 1. We first solve the homogeneous equation

$$y'' - 5y' + 6y = 0$$

by considering the characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0 \implies \lambda = \frac{5}{2} \pm \frac{1}{2} \implies \lambda = 2, 3.$$

This leads the general homogeneous solution $y_h(x) = Ae^{2x} + Be^{3x}$. For the particular solution, we make the ansatz $y_p(x) = Ce^{-2x}$ and we fix the constant C by considering

$$y_p'' - 5y' + 6y = 20Ce^{-2x} = e^{-2x} \implies C = \frac{1}{20}.$$

To fix the constants A, B we consider the initial conditions as

$$A + B + \frac{1}{20} = 1,$$
$$2A + 3B - \frac{1}{10} = 0$$

which has the solution $A = \frac{11}{4}, B = \frac{-9}{5}$. Consequently the full solution can be written as

$$y(x) = y_h(x) + y_p(x) = \frac{11}{4}e^{2x} - \frac{9}{5}e^{3x} + \frac{1}{20}e^{-2x}.$$

2. To find the power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ of the ODE $y'' - 2xy' - 2y = 0$ with initial conditions corresponding to $a_0 = 1, a_1 = 0$ we plug in the power series to the ODE to find

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x_n$$
$$= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty} 2na_n x^n - \sum_{n=0}^{\infty} 2a_n x_n = 0.$$

For this to be true, we need to have

$$(n+1)(n+2)a_{n+2} = 2a_n(n+1) \implies a_{n+2} = a_n\frac{2}{n+2}.$$

From $a_1 = 0$ we see that all odd indices of a_n must be zero, i.e., $a_{2n+1} = 0$ for all n. For the even indices, we compute

$$a_0 = 1$$
$$a_2 = \frac{2^1}{2}$$
$$a_4 = \frac{2^2}{2 \cdot 4}$$
$$a_6 = \frac{2^3}{2 \cdot 4 \cdot 6}$$
$$a_8 = \frac{2^4}{2 \cdot 4 \cdot 6 \cdot 8}$$
$$\vdots$$
$$a_{2n} = \frac{2^n}{2^n \cdot n!} = \frac{1}{n!}$$

Plugging this into the power series (and taking into account that we only want the even terms) we get

$$y(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} = \sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = e^{x^2}.$$