EXERCISES 9.6

Find Maclaurin series representations for the functions in Exercises 1–14. For what values of x is each representation valid?

1. e^{3x+1}

2. $\cos(2x^3)$

3. $\sin(x - \pi/4)$

4. $\cos(2x - \pi)$

5. $x^2 \sin(x/3)$

6. $\cos^2(x/2)$

7. $\sin x \cos x$

8. $\tan^{-1}(5x^2)$

9. $\dfrac{1+x^3}{1+x^2}$

10. $\ln(2 + x^2)$

11. $\ln \dfrac{1+x}{1-x}$

12. $(e^{2x^2} - 1)/x^2$

13. $\cosh x - \cos x$

14. $\sinh x - \sin x$

Find the required Taylor series representations of the functions in Exercises 15–26. Where is each series representation valid?

15. $f(x) = e^{-2x}$ about -1

16. $f(x) = \sin x$ about $\pi/2$

17. $f(x) = \cos x$ in powers of $x - \pi$

18. $f(x) = \ln x$ in powers of $x - 3$

19. $f(x) = \ln(2 + x)$ in powers of $x - 2$

20. $f(x) = e^{2x+3}$ in powers of $x + 1$

21. $f(x) = \sin x - \cos x$ about $\dfrac{\pi}{4}$

22. $f(x) = \cos^2 x$ about $\dfrac{\pi}{8}$

23. $f(x) = 1/x^2$ in powers of $x + 2$

24. $f(x) = \dfrac{x}{1+x}$ in powers of $x - 1$

25. $f(x) = x \ln x$ in powers of $x - 1$

26. $f(x) = xe^x$ in powers of $x + 2$

Find the first three nonzero terms in the Maclaurin series for the functions in Exercises 27–30.

27. $\sec x$

28. $\sec x \tan x$

29. $\tan^{-1}(e^x - 1)$

30. $e^{\tan^{-1} x} - 1$

31. Use the fact that $(\sqrt{1+x})^2 = 1 + x$ to find the first three nonzero terms of the Maclaurin series for $\sqrt{1+x}$.

32. Does $\csc x$ have a Maclaurin series? Why? Find the first three nonzero terms of the Taylor series for $\csc x$ about the point $x = \pi/2$.

Find the sums of the series in Exercises 33–36.

33. $1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dfrac{x^8}{4!} + \cdots$

34. $x^3 - \dfrac{x^9}{3! \times 4} + \dfrac{x^{15}}{5! \times 16} - \dfrac{x^{21}}{7! \times 64} + \dfrac{x^{27}}{9! \times 256} - \cdots$

35. $1 + \dfrac{x^2}{3!} + \dfrac{x^4}{5!} + \dfrac{x^6}{7!} + \dfrac{x^8}{9!} + \cdots$

36. $1 + \dfrac{1}{2 \times 2!} + \dfrac{1}{4 \times 3!} + \dfrac{1}{8 \times 4!} + \cdots$

37. Let $P(x) = 1 + x + x^2$. Find (a) the Maclaurin series for $P(x)$ and (b) the Taylor series for $P(x)$ about 1.

38. Verify by direct calculation that $f(x) = 1/x$ is analytic at a for every $a \neq 0$.

39. Verify by direct calculation that $\ln x$ is analytic at a for every $a > 0$.

40. Review Exercise 41 of Section 4.5. It shows that the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has derivatives of all orders at every point of the real line, and $f^{(k)}(0) = 0$ for every positive integer k. What is the Maclaurin series for $f(x)$? What is the interval of convergence of this Maclaurin series? On what interval does the series converge to $f(x)$? Is f analytic at 0?

41. By direct multiplication of the Maclaurin series for e^x and e^y show that $e^x e^y = e^{x+y}$.

EXERCISES 9.8

Find Maclaurin series representations for the functions in Exercises 1–8. Use the binomial series to calculate the answers.

1. $\sqrt{1+x}$

2. $x\sqrt{1-x}$

3. $\sqrt{4+x}$

4. $\dfrac{1}{\sqrt{4+x^2}}$

5. $(1-x)^{-2}$

6. $(1+x)^{-3}$

7. $\cos^{-1} x$

8. $\sinh^{-1} x$

9. **(Binomial coefficients)** Show that the binomial coefficients $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ satisfy

(i) $\binom{n}{0} = \binom{n}{n} = 1$ for every n, and

(ii) if $0 \le k \le n$, then $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

It follows that, for fixed $n \ge 1$, the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

are the elements of the nth row of **Pascal's triangle** below, where each element with value greater than 1 is the sum of the two diagonally above it.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \cdot & & & & & & & & \cdot & \\
 & & \cdot & & & & & & \cdot & &
\end{array}$$

10. **(An inductive proof of the Binomial Theorem)** Use mathematical induction and the results of Exercise 9 to prove the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$= a^n + na^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \cdots + b^n.$$

11. **(The Leibniz Rule)** Use mathematical induction, the Product Rule, and Exercise 9 to verify the Leibniz Rule for the nth derivative of a product of two functions:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$
$$= f^{(n)} g + n f^{(n-1)} g' + \binom{n}{2} f^{(n-2)} g''$$
$$+ \binom{n}{3} f^{(n-3)} g^{(3)} + \cdots + f g^{(n)}.$$

12. **(Proof of the Multinomial Theorem)** Use the Binomial Theorem and induction on n to prove Theorem 24. *Hint:* Assume the theorem holds for specific n and all k. Apply the Binomial Theorem to

$$(x_1 + \cdots + x_n + x_{n+1})^k = \big((x_1 + \cdots + x_n) + x_{n+1}\big)^k.$$

13. **(A Multifunction Leibniz Rule)** Use the technique of Exercise 12 to generalize the Leibniz Rule of Exercise 11 to calculate the kth derivative of a product of n functions $f_1 f_2 \cdots f_n$.