Introduction to conic sections

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1 Introduction

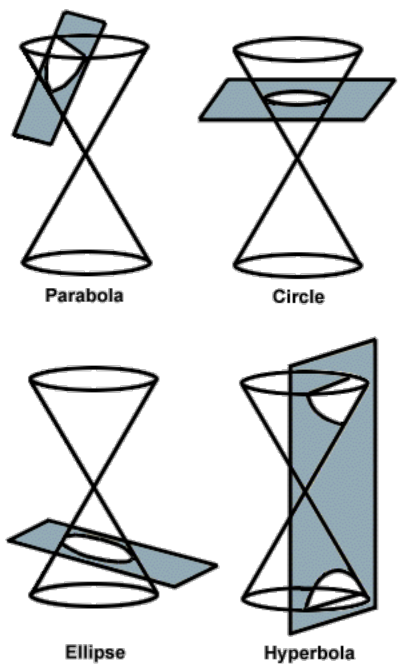
A conic is a two-dimensional figure created by the intersection of a plane and a right circular cone. All conics can be written in terms of the following equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\,.$$

The four conics we'll explore in this text are parabolas, ellipses, circles, and hyperbolas. The equations for each of these conics can be written in a standard form, from which a lot about the given conic can be told without having to graph it. We'll study the standard forms and graphs of these four conics,

1.1 General definition

A conic is the intersection of a plane and a right circular cone. The four basic types of conics are parabolas, ellipses, circles, and hyperbolas. Study the figures below to see how a conic is geometrically defined.



In a *non-degenerate conic* the plane does not pass through the vertex of the cone. When the plane does intersect the vertex of the cone, the resulting conic is called a *degenerate conic*. Degenerate conics include a point, a line, and two intersecting lines.

The equation of every conic can be written in the following form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\,.$$

This is the algebraic definition of a conic. Conics can be classified according to the coefficients of this equation.

The *discriminant* of the equation is $B^2 - 4AC$. Assuming a conic is not degenerate, the following conditions hold true:

1. If $B^2 - 4AC < 0$, the conic is a *circle* (if $B = 0$ and $A = B$), or an *ellipse*.

2. If $B^2 - 4AC = 0$, the conic is a parabola.

3. If $B^2 - 4AC > 0$, the conic is a *hyperbola*.

Although there are many equations that describe a conic section, the following table gives the *standard form* equations for non-degenerate conics sections.

Standard equation for non-degenerate conic section	
circle	$x^2 + y^2 = a^2$
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
parabola	$y^2 - 4ax = 0$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

1.2 problems

1. Is the following conic a parabola, an ellipse, a circle, or a hyperbola: $-3x^2 + y + 2 = 0$? It is a parabola.

2. Is the following conic a parabola, an ellipse, a circle, or a hyperbola: $2x^2 + 3xy - 4y^2 + 2x - 3y + 1 = 0$? It is a hyperbola.

3. Is the following conic a parabola, an ellipse, a circle, or a hyperbola: $2x^2 - 3y^2 = 0$? It is a hyperbola.

4. Is the following conic a parabola, an ellipse, a circle, or a hyperbola: $-3x^2 + xy - 2y^2 + 4 = 0$? It is an ellipse.

5. Is the following conic a parabola, an ellipse, a circle, or a hyperbola: $x^2 = 0$? It is a degenerate conic. $x = 0$ is a line.

6. Is the following conic a parabola, an ellipse, a circle, or a hyperbola: $x^2 - y^2 = 0$? It is a degenerate conic. $x^2 - y^2 = (x - y)(x + y) = 0$ are two lines that intersects.

7. Is the following conic a parabola, an ellipse, a circle, or a hyperbola: $x^2 + y^2 = 0$? It is a degenerate conic. The only point that satisfies the equations $x^2 + y^2 = 0$ is $(0, 0)$.

1.3 Geometric definition

Let ε be a positive number, *eccentricity*, ℓ a line, *directice* and a point $\mathcal{B}$, *focus.* The triple $(\varepsilon, \ell, \mathcal{B})$ defines a conic section in the following way:

A point $\mathcal{P}$ is in the conic section defined by $(\varepsilon, \ell, \mathcal{B})$ if

$$|\mathcal{P}\mathcal{B}| = \varepsilon \cdot |\mathcal{P}\ell|$$

$|\mathcal{P}\mathcal{B}|$ stands for the distance from the point $\mathcal{P}$ to the point $\mathcal{B}$ and $|\mathcal{P}\ell|$ for the minimal distance of the point $\mathcal{P}$ to the line ℓ.

If the focus $\mathcal{B}$ does not belong to the directrice line ℓ, the following conditions hold true:

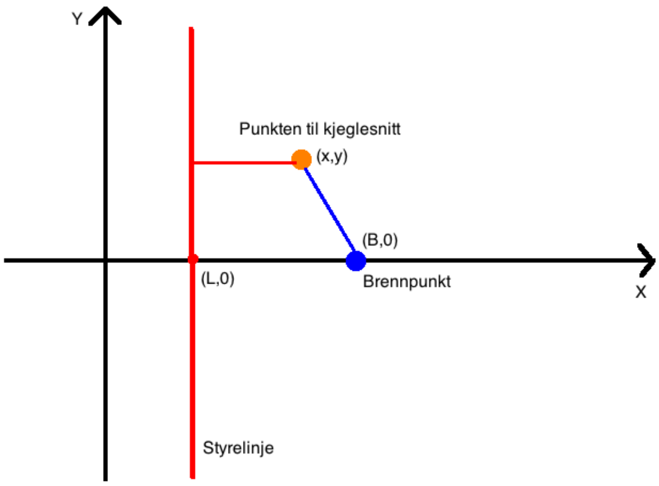
1. If $0 < \varepsilon < 1$ then conic is an ellipse.
2. If $\varepsilon = 1$ then conic is an parabola.
3. If $\varepsilon > 1$ then conic is an hyperbola,

If the focus $\mathcal{B}$ does belong to the directrice line ℓ, the following conditions hold true:

1. If $0 < \varepsilon < 1$ then conic is a point.
2. If $\varepsilon = 1$ then conic is a line.
3. If $\varepsilon > 1$ then conic are two lines that cross.

2 Non-degenerate conic sections

Given an eccentricity ε, a directrice line ℓ and a focus point $\mathcal{B}$ not contain in ℓ, we can define a non-degenerate conic section. For simplicity we will assume that ℓ is of the form $x = L$ and $\mathcal{B} = (B, 0)$, with $L < B$. We will see later that through translations and rotations we always can reduce to this situation.



In this case given a point $\mathcal{P} = (x, y)$ we have that

$$|\mathcal{P}\mathcal{B}| = \sqrt{(x-B)^2 + y^2} \qquad \textit{and} \qquad |\mathcal{P}\ell| = \sqrt{(x-L)^2}.$$

Then the relation $|\mathcal{P}\mathcal{B}| = \varepsilon \cdot |\mathcal{P}\ell|$ can we written in the following way:

$$\sqrt{(x-B)^2 + y^2} = \varepsilon\sqrt{(x-L)^2}.$$

Then we have

$$(\sqrt{(x-B)^2 + y^2})^2 = (\varepsilon\sqrt{(x-L)^2})^2$$

that is equivalent to

$$\boxed{(x-B)^2 + y^2 = \varepsilon^2(x-L)^2\,.}$$

So this is the *general equation of a conic section*. Now we will study which type of conic section is depending of the possible values of the eccentricity ε.

2.1 Ellipse

We suppose that $0 < \varepsilon < 1$. First we compute the intersection of the conic section with the x-axis. To do that we have to replace $y = 0$ in the general equation of the conic section, so it follows the equation

$$(x-B)^2 = \varepsilon^2(x-L)^2\,.$$

This is equivalent to the equation

$$\sqrt{(x-B)^2} = \pm\sqrt{\varepsilon^2(x-L)^2}\,,$$

so we have

$$(x - B) = \pm\varepsilon(x - L)\,,$$

Here we encounter to possibilities: First suppose the equation

$$(x - B) = -\varepsilon(x - L)\,,$$

that is equivalent to

$$(1+\varepsilon)x = B + \varepsilon L\,,$$

so the first point that intersects the x-axis is

$$\boxed{x_1 = x = \tfrac{B+\varepsilon L}{1+\varepsilon}\,.}$$

Finally suppose the equation

$$(x - B) = +\varepsilon(x - L)\,,$$

that is equivalent to

$$(1-\varepsilon)x = B - \varepsilon L\,,$$

so the second point that intersects the x-axis is

$$\boxed{x_2 = x = \tfrac{B-\varepsilon L}{1-\varepsilon}\,.}$$

A simple calculation yields that $x_1 < x_2$.

definitions	
center	$\bar{x} = \frac{x_1+x_2}{2}$
major axis	$a = \frac{x_2-x_1}{2}$
minor axis	$b = a\sqrt{1-\varepsilon^2}$

With this definitions on hand we can rewrite the general equation in the following way

$$\boxed{\tfrac{(x-\bar{x})^2}{a^2} + \tfrac{y^2}{b^2} = 1\,,}$$

that we call *the standard equation of the ellipse.*

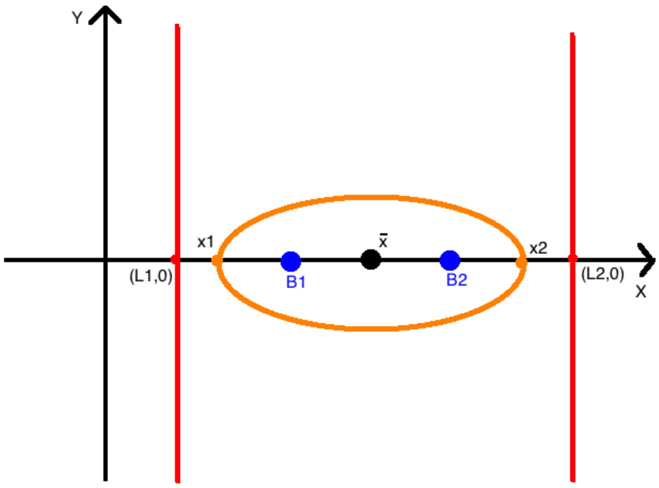
Conversely,

$\frac{(x-\bar{x})^2}{a^2} + \frac{y^2}{b^2} = 1$	
eccentricity	$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$
directrice	$L = \bar{x} - \frac{a}{\varepsilon}$
focus	$B = \bar{x} - \varepsilon \cdot a$

From the standard equation of the ellipse one can observe that the ellipse is symmetric with respect to the vertical line $x = \bar{x}$. Therefore if we define

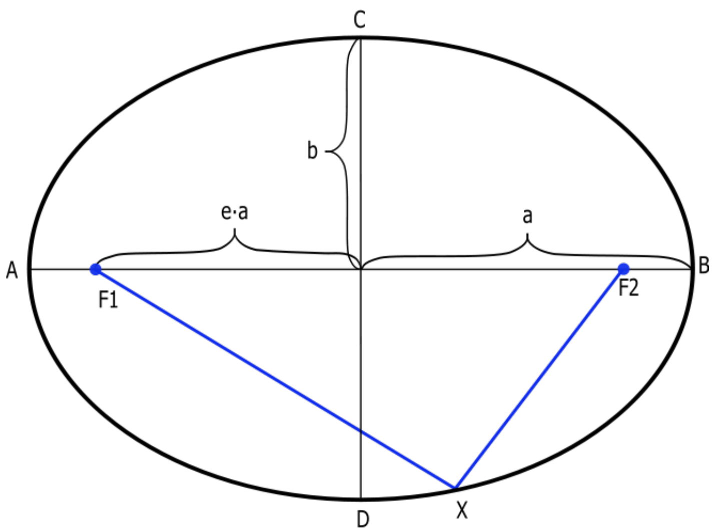
$$B_2 = \bar{x} + \frac{a}{\varepsilon} \qquad \text{and} \qquad \mathrm{L}_2 = \bar{\mathrm{x}} + \frac{\mathrm{a}}{\varepsilon}$$

we have that the triple given by eccentricity ε, focus point $\mathcal{B}_2 = (B_2, 0)$ and the directrice line ℓ_2 given by $x = L_2$, determines the same conic section as $(\varepsilon, \mathcal{B}, \ell)$. Thus, $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_2$ are called the two focus points of the ellipse.



Now given the two focus of the ellipse $\mathcal{B}_1$ and $\mathcal{B}_2$ we can give an alternative geometric description, in the following way: An ellipse is the set of points such that the sum of the distances from any point on the ellipse to $\mathcal{B}_1$ and $\mathcal{B}_2$ is constant and equal to $2a$, that is

$$|\mathcal{P}\mathcal{B}_1| + |\mathcal{P}\mathcal{B}_2| = 2a$$



2.1.1 Examples

1. Find the equation of the ellipse with eccentricity $\varepsilon = 1/3$, directrice line $x = -1$ and focus $\mathcal{B} = (1, 0)$. Then according the formulas we have

$$x_1 = \frac{1 + 1/3(-1)}{1 + 1/3} = \frac{2/3}{4/3} = 1/2 \qquad x_2 = \frac{1 - 1/3(-1)}{1 - 1/3} = \frac{4/3}{2/3} = 2$$

and hence the center of the ellipse is

$$\bar{x} = \frac{1/2 + 2}{2} = 5/4\,,$$

and

$$a = \frac{2 - 1/2}{2} = 3/4 \qquad b = 3/4 \cdot \sqrt{1 - (1/3)^2} = 3/4 \cdot \sqrt{8/9} = \frac{\sqrt{2}}{2}\,.$$

Therefore the equation is

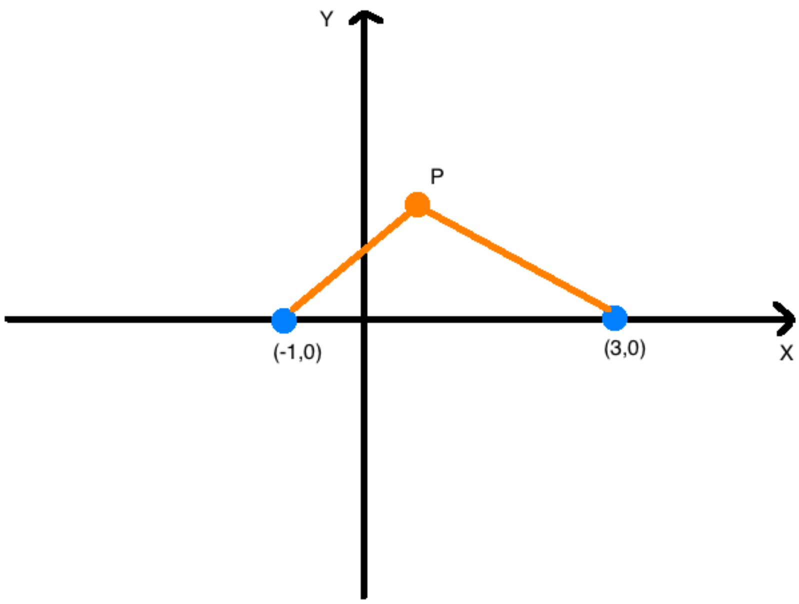
$$\frac{(x - 5/4)^2}{(3/4)^2} + \frac{y^2}{(\frac{\sqrt{2}}{2})^2} = 1\,,$$

so

$$\boxed{\frac{16(x - 5/4)^2}{9} + 2y^2 = 1}$$

2. Let $\mathcal{B}_1 = (-1, 0)$ and $\mathcal{B}_2 = (3, 0)$ be two points in the plane. We want to give the equation of the ellipse such points $\mathcal{P}$ satisfy

$$|\mathcal{P}\mathcal{B}_1| + |\mathcal{P}\mathcal{B}_2| = 6\,.$$



First observe that the formulas say that $2a = 6$, and hence $a = 3$. The center of the ellipse is the mid-point between $\mathcal{B}_1$ and $\mathcal{B}_2$ that is $\bar{x} = 2$. We need now to calculate the eccentricity, that from the above formulas comes from the relation

$$|\mathcal{B}_1\mathcal{B}_2| = 2a\varepsilon\,,$$

so we have that $4 = 2 \cdot 3 \cdot \varepsilon$, and follows that $\varepsilon = 2/3$. Finally, we have that $b = a\sqrt{1-\varepsilon^2}$, so $b = 3\sqrt{5/9} = \sqrt{5}$. Therefore the equation of the ellipse is

$$\boxed{\frac{(x-2)^2}{9} + \frac{y^2}{5} = 1}$$

2.2 Parabel

We suppose that $\varepsilon = 1$, that translates as the condition

$$|\mathcal{PB}| = \varepsilon \cdot |\mathcal{P}\ell|\,,$$

that are the points $\mathcal{P}$ in the plane that are at the same distance from the focus $\mathcal{B}$ as from the directrice ℓ.

Then the general equation of the conic section reduces to

$$(x-B)^2 + y^2 = (x-L)^2\,,$$

and we can write it as

$$y^2 = (x-L)^2 - (x-B)^2 = x^2 - 2xL + L^2 - x^2 + 2xB - B^2 = 2(B-L)x + (L^2 - B^2)\,,$$

that is

$$\boxed{y^2 = 2(B-L)x + (L^2 - B^2)}$$

If we want to find the intersection of the conic section with the x-axis, we have to replace $y = 0$ in the above equation. So we have

$$0 = 2(B-L)x + (L^2 - B^2)$$

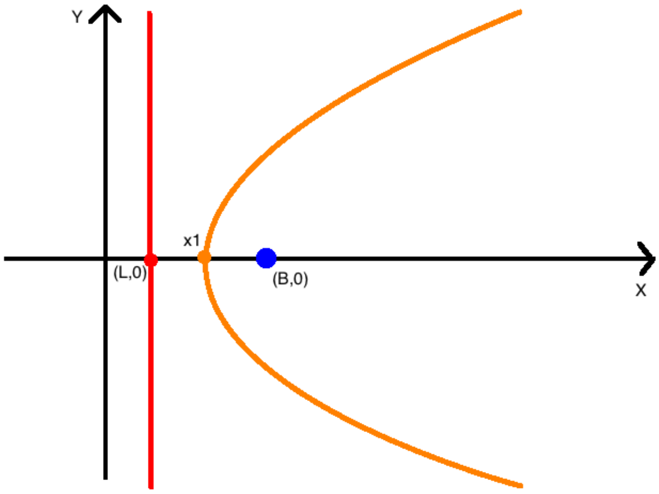
that is

$$2(L-B)x = (L^2 - B^2) = (L-B)(L+B)$$

so after cancel out some the $(L-B)$ term we have that

$$\boxed{x_1 = x = \frac{L+B}{2}}$$

that we can the *vertex* of the parabola.



2.3 Hyperbola

We suppose that $\varepsilon > 1$. First we compute the intersection of the conic section with the x-axis. To do that we have to replace $y = 0$ in the general equation of the conic section, so it follows the equation

$$(x - B)^2 = \varepsilon^2 (x - L)^2 .$$

This is equivalent to the equation

$$\sqrt{(x - B)^2} = \pm\sqrt{\varepsilon^2 (x - L)^2} ,$$

so we have

$$(x - B) = \pm\varepsilon(x - L) ,$$

Here we encounter to possibilities: First suppose the equation

$$(x - B) = -\varepsilon(x - L) ,$$

that is equivalent to

$$(1 + \varepsilon)x = B + \varepsilon L ,$$

so the first point that intersects the x-axis is

$$\boxed{x_1 = x = \tfrac{B+\varepsilon L}{1+\varepsilon} .}$$

Finally suppose the equation

$$(x - B) = +\varepsilon(x - L) ,$$

that is equivalent to

$$(1-\varepsilon)x = B - \varepsilon L\,,$$

so the second point that intersects the x-axis is

$$\boxed{x_2 = x = \frac{B-\varepsilon L}{1-\varepsilon}\,.}$$

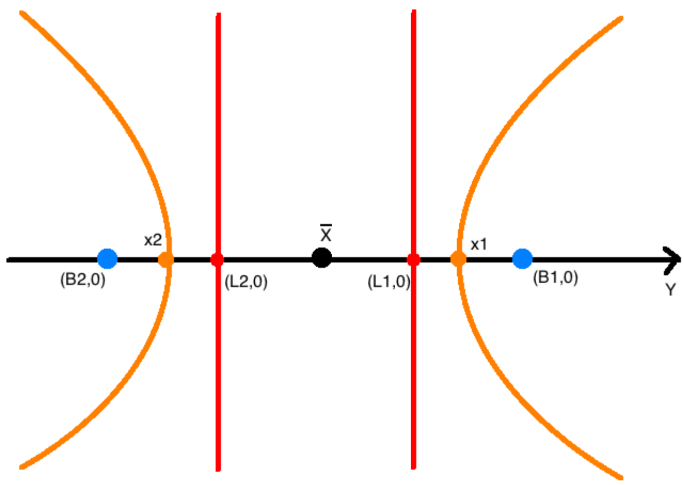
A simple calculation yields that $x_1 > x_2$. Observe that this is the opposite that happens in the ellipse situation.

definitions	
center	$\bar{x} = \frac{x_1+x_2}{2}$
major axis	$a = \frac{x_1-x_2}{2}$
minor axis	$b = a\sqrt{\varepsilon^2-1}$

With this definitions on hand we can rewrite the general equation in the following way

$$\boxed{\frac{(x-\bar{x})^2}{a^2} - \frac{y^2}{b^2} = 1\,,}$$

that we call *the standard equation of the hyperbola.*



Conversely,

$\frac{(x-\bar{x})^2}{a^2} - \frac{y^2}{b^2} = 1$	
eccentricity	$\varepsilon = \sqrt{1+\frac{b^2}{a^2}}$
directrice	$L = \bar{x} + \frac{a}{\varepsilon}$
focus	$B = \bar{x} + \varepsilon \cdot a$

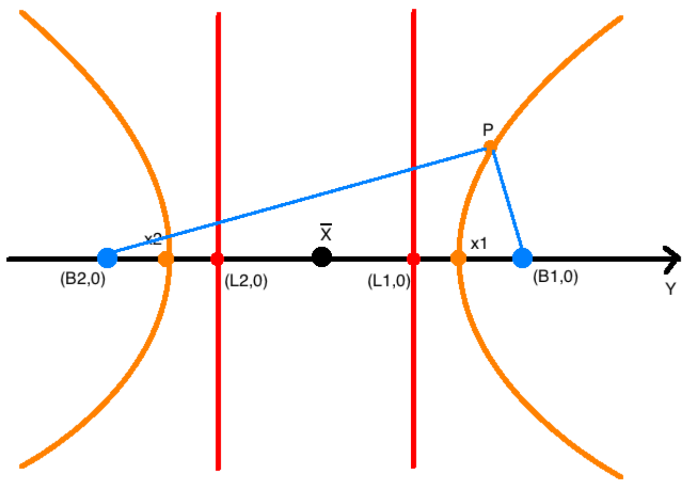
From the standard equation of the hyperbola one can observe that the hyperbola is symmetric with respect to the vertical line $x = \bar{x}$. Therefore if we define

$$B_2 = \bar{x} - \varepsilon \cdot a \qquad \text{and} \qquad \mathrm{L}_2 = \bar{\mathrm{x}} - \frac{\mathrm{a}}{\varepsilon}$$

we have that the triple given by eccentricity ε, focus point $\mathcal{B}_2 = (B_2, 0)$ and the directrice line ℓ_2 given by $x = L_2$, determines the same conic section as $(\varepsilon, \mathcal{B}, \ell)$. Thus, $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_2$ are called the two focus points of the hyperbola.

Now given the two focus of the hyperbola $\mathcal{B}_1$ and $\mathcal{B}_2$ we can give an alternative geometric description, in the following way: An hyperbola is the set of points such that the difference of the distances from any point on the ellipse to $\mathcal{B}_1$ and $\mathcal{B}_2$ is constant and equal to $2a$, that is

$$|\mathcal{P}\mathcal{B}_1| - |\mathcal{P}\mathcal{B}_2| = 2a \qquad \text{or} \qquad |\mathcal{P}\mathcal{B}_2| - |\mathcal{P}\mathcal{B}_1| = 2\mathrm{a}\,.$$



2.3.1 Examples

1. Find the equation of the hyperbola with eccentricity $\varepsilon = 2$, directrice line $x = -1$ and focus $\mathcal{B} = (1, 0)$. Then according the formulas we have

$$x_1 = \frac{1 + 2(-1)}{1 + 2} = -\frac{1}{3} \qquad x_2 = \frac{1 - 2(-1)}{1 - 2} = \frac{3}{-1} = -3$$

and hence the center of the hyperbola is

$$\bar{x} = \frac{-1/3 - 3}{2} = -5/3\,,$$

and

$$a = \frac{-1/3 - (-3)}{2} = 4/3 \qquad b = 4/3 \cdot \sqrt{2^2 - 1} = 4/3 \cdot \sqrt{3} = \frac{4}{\sqrt{3}}\,.$$

Therefore the equation is

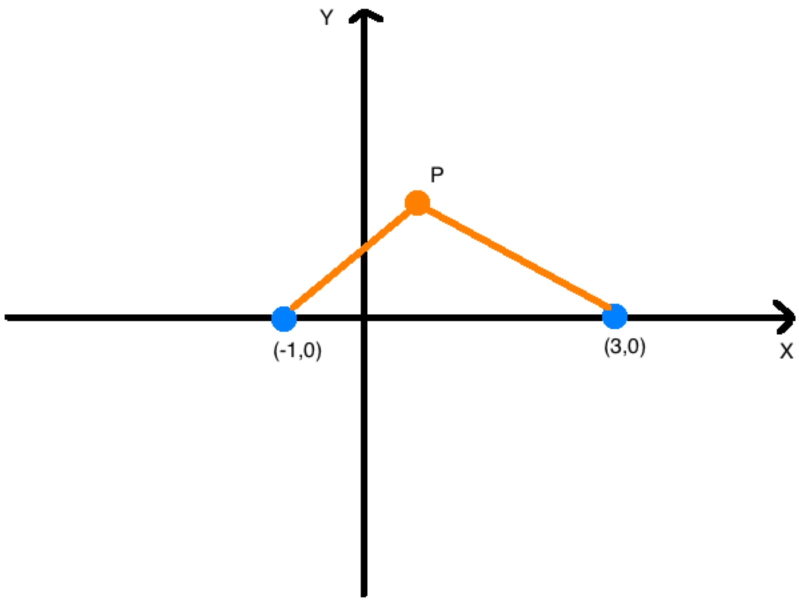
$$\frac{(x+5/3)^2}{(4/3)^2} - \frac{y^2}{(\frac{4}{\sqrt{3}})^2} = 1\,,$$

so

$$\boxed{\frac{9(x+5/3)^2}{16} - \frac{3y^2}{16} = 1}$$

2. Let $\mathcal{B}_1 = (-1, 0)$ and $\mathcal{B}_2 = (3, 0)$ be two points in the plane. We want to give the equation of the hyperbola such points $\mathcal{P}$ satisfy

$$|\mathcal{P}\mathcal{B}_1| - |\mathcal{P}\mathcal{B}_2| = 6 \qquad \text{or} \qquad |\mathcal{P}\mathcal{B}_2| - |\mathcal{P}\mathcal{B}_1| = 6\,.$$



First observe that the formulas say that $2a = 6$, and hence $a = 3$. The center of the ellipse is the mid-point between $\mathcal{B}_1$ and $\mathcal{B}_2$ that is $\bar{x} = 2$. We need now to calculate the eccentricity, that from the above formulas comes from the relation

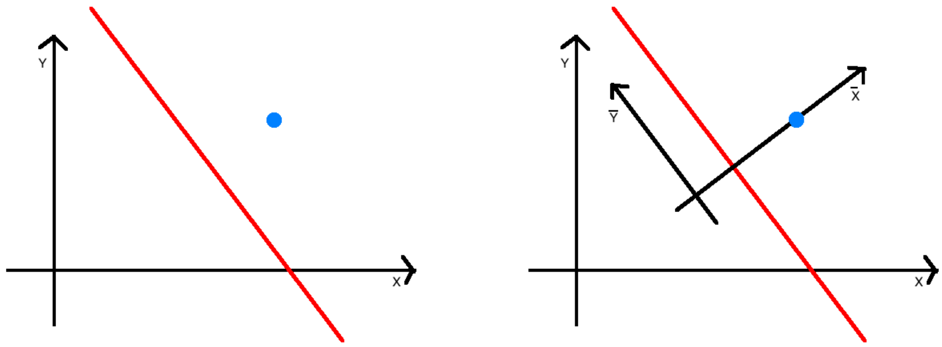
$$|\mathcal{B}_1\mathcal{B}_2| = \frac{2a}{\varepsilon}\,,$$

so we have that $4 = \frac{2\cdot 3}{\varepsilon}$, and follows that $\varepsilon = 3/2$. Finally, we have that $b = a\sqrt{\varepsilon^2 - 1}$, so $b = 3\sqrt{5/4} = \frac{3\sqrt{5}}{2}$. Therefore the equation of the hyperbola is

$$\boxed{\frac{(x-2)^2}{9} - \frac{4y^2}{45} = 1}$$

3 Change of coordinates

In the above section we have supposed that the directrice line is parallel to the y-axis, i.e., $x = L$ and the focus is over the $x-$axis, i.e. $\mathcal{B} = (B, 0)$, but what happens with the general situation where we have any given line and point? Change of coordinates!

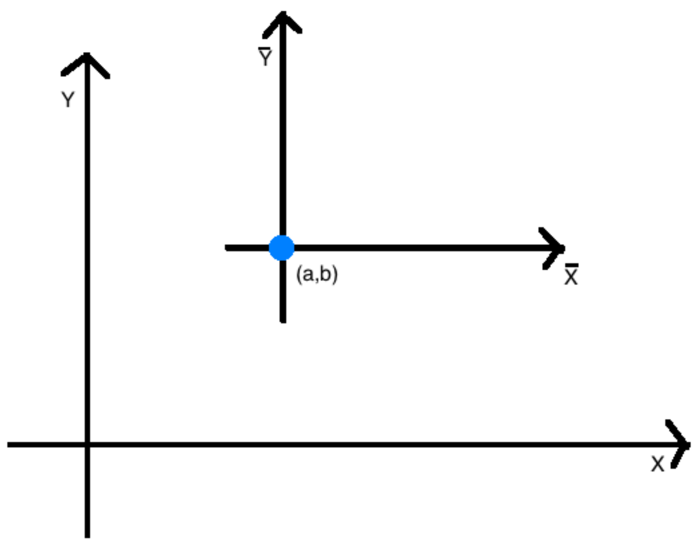


3.1 translation

A *translation to a point* (a, b) is a change of coordinates (x, y) to a new coordinates $(\bar{x}, \bar{y})$ in such a way

$$\bar{x} = x - a \qquad \text{and} \qquad \bar{\mathrm{y}} = \mathrm{y} - \mathrm{b}\,.$$

Roughly speaking, a translation moves the origin to the point (a, b).



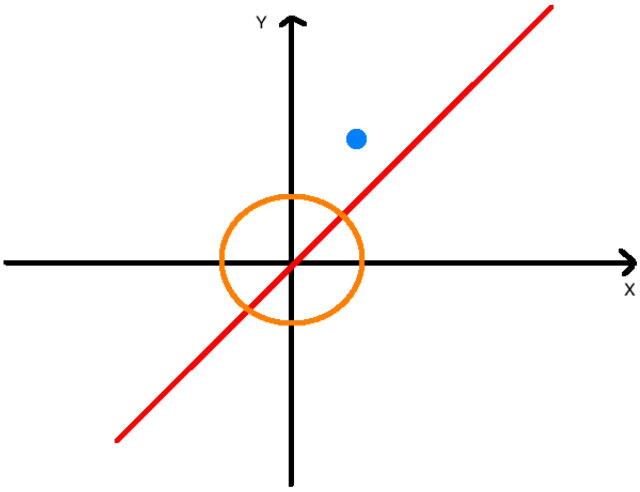
We can reverse the change of coordinates from the new coordinates $(\bar{x}, \bar{y})$ to the old ones:

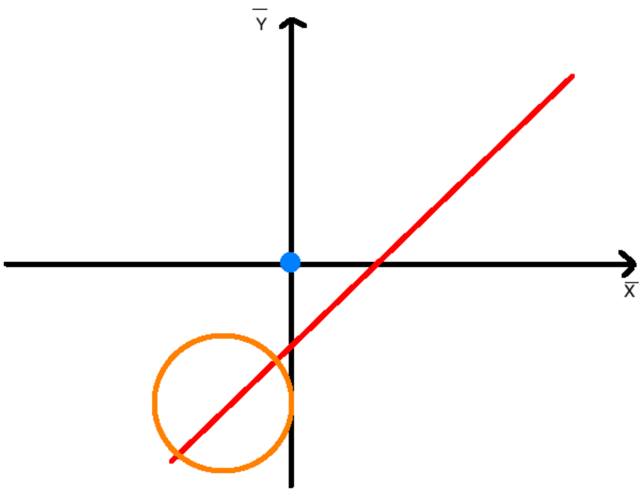
$$x = \bar{x} + a \qquad \text{and} \qquad \mathrm{y} = \bar{\mathrm{y}} + \mathrm{b}\,.$$

3.1.1 Examples

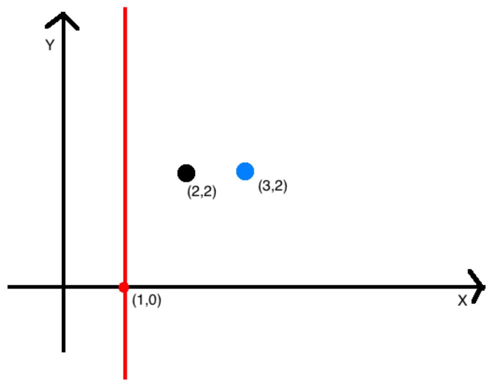
1. We consider the translation to the point $(1, 2)$. Then:

(x, y)-coordinates	$(\bar{x}, \bar{y})$-coordinates
$(0, 0)$	$(-1, -2)$
$(1, 2)$	$(0, 0)$
$y = x$	$\bar{y} + 2 = \bar{x} + 1$ $\bar{y} = \bar{x} - 1$
$x^2 + y^2 = 1$	$(\bar{x} + 1)^2 + (\bar{y} + 2)^2 = 1$





2. We want to find the equation of the ellipse that has eccentricity $\varepsilon = 1/2$, directrice line $x = 1$ and focus $(3, 2)$.



Observe that if we make a translation to the point $(2, 2)$ we have the following

(x, y)-coordinates	$(\bar{x}, \bar{y})$-coordinates
$(3, 2)$	$(1, 0)$
$x = 1$	$\bar{x} + 2 = 1$ $\bar{x} = -1$

So now we can construct the ellipse with eccentricity $\varepsilon = 1/2$, directrice line $\bar{x} = -1 = L$ and focus $(1, 0)$, so $B = 1$. According the formulas we have that

$$x_1 = \frac{1 + 1/2 \cdot (-1)}{1 + 1/2} = \frac{1/2}{3/2} = 1/3$$

and

$$x_2 = \frac{1 - 1/2 \cdot (-1)}{1 - 1/2} = \frac{3/2}{1/2} = 3\,.$$

thus

$$\bar{x} = \frac{1/3 + 3}{2} = \frac{10/3}{2} = 5/3$$

$$a = \frac{3 - 1/3}{2} = \frac{8/3}{2} = 4/3$$

and

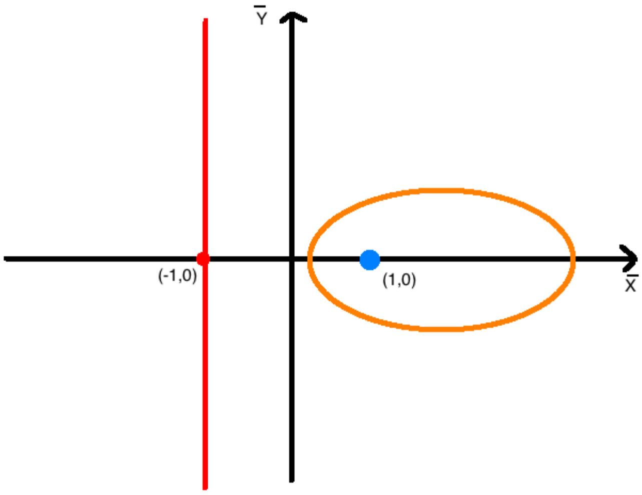
$$b = 4/3 \cdot \sqrt{1 - (1/2)^2} = 4/3 \cdot \sqrt{3/4} = \frac{2}{\sqrt{3}}\,.$$

Therefore the equation of the ellipse in the $(\bar{x}, \bar{y})$ coordinates is

$$\frac{(\bar{x} - 5/3)^2}{(4/3)^2} + \frac{\bar{y}^2}{(\frac{2}{\sqrt{3}})^2} = 1$$

that we can rewrite as

$$\frac{(\bar{x}-5/3)^2}{16/9}+\frac{\bar{y}^2}{4/3}=1\,.$$



Finally we return to the old coordinates (x, y), using that

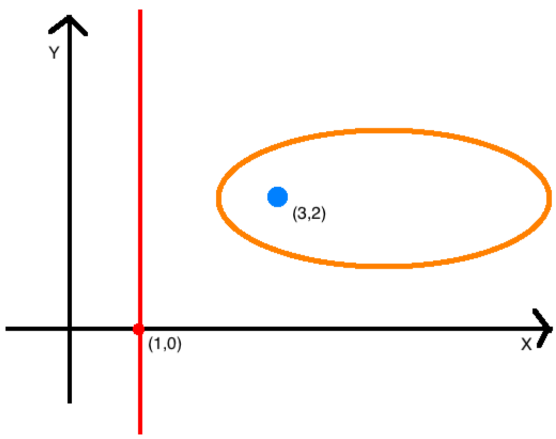
$$\bar{x}=x-2 \qquad \text{and} \qquad \bar{\mathrm{y}}=\mathrm{y}-2\,.$$

So replacing this to the equation we have

$$\frac{((x-2)-5/3)^2}{16/9}+\frac{(y-2)^2}{4/3}=1\,.$$

that is

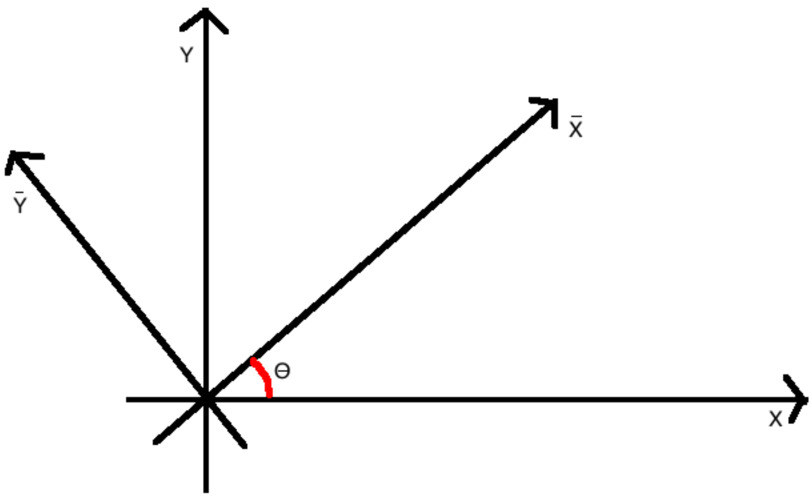
$$\boxed{\frac{(x-11/3)^2}{16/9}+\frac{(y-2)^2}{4/3}=1\,.}$$



3.2 rotation

A *rotation with angle* θ is a change of coordinates (x, y) to a new coordinates $(\bar{x}, \bar{y})$ in such a way

$$\bar{x} = x\cos\theta + y\sin\theta \qquad \text{and} \qquad \bar{y} = -x\sin\theta + y\cos\theta\,.$$



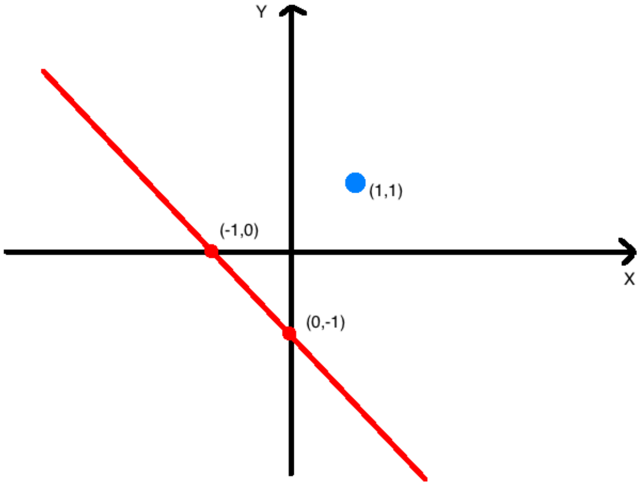
We can reverse the change of coordinates from the new coordinates $(\bar{x}, \bar{y})$ to the old ones:

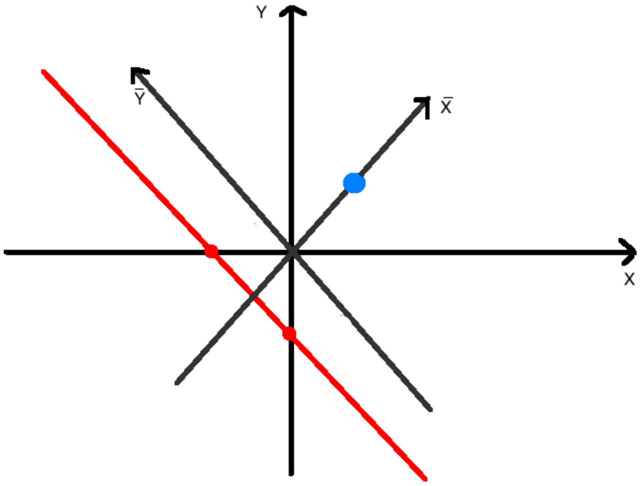
$$x = \bar{x}\cos\theta - \bar{y}\sin\theta \qquad \text{and} \qquad y = \bar{x}\sin\theta + \bar{y}\cos\theta\,.$$

3.2.1 Examples

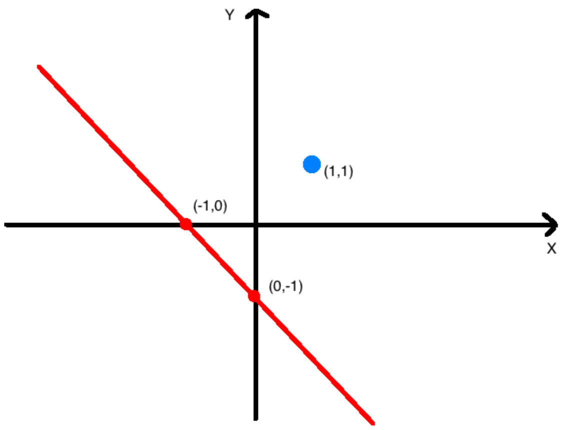
1. We consider the a rotation of 45°. Then:

(x, y)-coordinates	$(\bar{x}, \bar{y})$-coordinates
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(\sqrt{2}, 0)$
$y = -x - 1$	$(\frac{\sqrt{2}}{2}\bar{x} + \frac{\sqrt{2}}{2}\bar{y}) = -(\frac{\sqrt{2}}{2}\bar{x} - \frac{\sqrt{2}}{2}\bar{y}) - 1$ $\bar{x} = -\frac{1}{\sqrt{2}}$



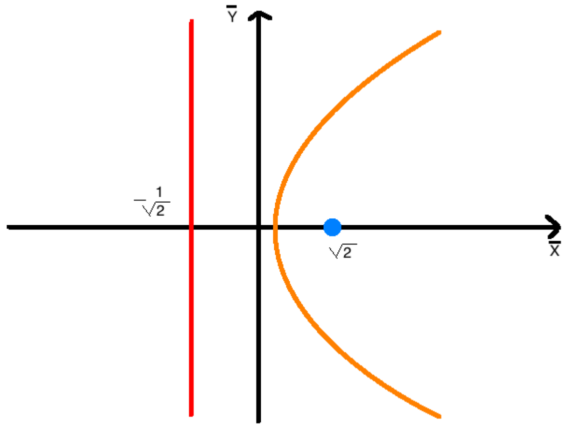


2. We want to find the equation of the parabola with directrice line $y = -x - 1$ and focus $(1, 1)$.



Observe that if we make a translation of 45° in the new coordinates $(\bar{x}, \bar{y})$ the directrice line has equation $\bar{x} = -\frac{1}{\sqrt{2}}$ and the focus $(\sqrt{2}, 0)$. Then we can write the equation of the parabola

$$\bar{y}^2 = 2(\sqrt{2} - (-\frac{1}{\sqrt{2}}))\bar{x} + ((-\frac{1}{\sqrt{2}})^2) - (\sqrt{2})^2 = \frac{3}{\sqrt{2}}\bar{x} - \frac{3}{2}$$



Finally, making the change of coordinate to the old coordinates (x, y), we have that

$$(-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y)^2 = \frac{3}{\sqrt{2}}(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y) - \frac{3}{2}$$

so it follows that

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 - xy = \frac{3}{2}x + \frac{3}{2}y - \frac{3}{2}$$

and hence the final equation is

$$\boxed{x^2 + y^2 - 2xy - 3x - 3y + 3 = 0}$$