NTNU
Kunnskap for en bedre verden

Institutt for matematiske fag

# Eksamensoppgave i **MA1101 Grunnkurs i analyse I**

**Faglig kontakt under eksamen:** Agamemnon Zafeiropoulos
**Tlf:**

**Eksamensdato:** 8 desember 2020
**Eksamenstid (fra–til):** 09:00 -13:00
**Hjelpemiddelkode/Tillatte hjelpemidler:**

**Annen informasjon:**

**Målform/språk:** bokmål
**Antall sider:** 10
**Antall sider vedlegg:** 0

**Kontrollert av:**

| **Informasjon om trykking av eksamensoppgave** |
|---|
| **Originalen er:** |
| **1-sidig** ☐ **2-sidig** ☒ |
| **sort/hvit** ☒ **farger** ☐ |
| **skal ha flervalgskjema** ☒ |

Dato Sign

**Oppgave 1**

Let

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x \sin x + e^x.$$

**a)** Find the third degree McLaurin polynomial of $f$.

**b)** Find the equation of the tangent line of the graph of $f$ at the point $(\pi, f(\pi))$.

**Solution:**
**a)** For any function $f$, the third degree McLaurin polynomial is

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 \cdot$$

For the given function $f$ we find

$$\begin{aligned} f'(x) &= \sin x + x\cos x + e^x, \\ f''(x) &= 2\cos x - x\sin x + e^x, \\ f^{(3)}(x) &= -3\sin x - x\cos x + e^x \end{aligned}$$

and therefore

$$f(0) = 1, f'(0) = 1, f''(0) = 3, f^{(3)}(0) = 1.$$

The McLaurin polynomial of $f$ is

$$P(x) = 1 + x + \frac{3}{2}x^2 + \frac{1}{6}x^3 \cdot$$

**b)** The equation of the tangent line at $(\pi, f(\pi))$ is

$$y - f(\pi) = f'(\pi)(x - \pi).$$

We have $f(\pi) = e^\pi$ and $f'(\pi) = e^\pi - \pi$, therefore the requested equation is

$$y = (e^\pi - \pi)(x - \pi) + e^\pi.$$

**Oppgave 2** Evaluate the following integrals.

$$\int_0^1 \arcsin^2 x \, \mathrm{d}x, \quad \int_0^2 \frac{x^2+2}{x+1} \mathrm{d}x, \quad \int_{-\pi}^{\pi} x^4 \sin x \, \mathrm{d}x.$$

**Solution:**

$$I_1 = \int_0^1 \arcsin^2 x \, \mathrm{d}x.$$

Set

$$u = \arcsin x \quad \Rightarrow \quad x = \sin u$$
$$\mathrm{d}x = \cos u \mathrm{d}u$$

$$x_1 = 0 \Rightarrow u_1 = 0, \qquad x_2 = 1 \Rightarrow u_2 = \frac{\pi}{2}.$$

Then

$$\begin{aligned} I_1 &= \int_0^{\frac{\pi}{2}} u^2 \cos u \mathrm{d}u = \int_0^{\frac{\pi}{2}} u^2 (\sin u)' \mathrm{d}u \\ &= \Big[u^2 \sin u\Big]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} 2u \sin u \mathrm{d}u \\ &= \frac{\pi^2}{4} + \int_0^{\frac{\pi}{2}} 2u (\cos u)' \mathrm{d}u \\ &= \frac{\pi^2}{4} + [2u \cos u]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} 2 \cos u \mathrm{d}u \\ &= \frac{\pi^2}{4} - 2. \end{aligned}$$

<u>Alternatively:</u> We have

$$\begin{aligned} I_1 &= \int_0^1 (x)' \arcsin^2 x \mathrm{d}x \\ &= \Big[x \arcsin^2 x\Big]_0^1 - \int_0^1 x \cdot \frac{2 \arcsin x}{\sqrt{1-x^2}} \mathrm{d}x \\ &= \frac{\pi^2}{4} + \int_0^1 (\sqrt{1-x^2})' \cdot 2 \arcsin x \, \mathrm{d}x \\ &= \frac{\pi^2}{4} + \Big[\sqrt{1-x^2} \cdot 2 \arcsin x\Big]_0^1 - \int_0^1 \sqrt{1-x^2} \cdot \frac{2}{\sqrt{1-x^2}} \mathrm{d}x \\ &= \frac{\pi^2}{4} - 2. \end{aligned}$$

For the second integral we set $u = x + 1$ so that $\mathrm{d}u = \mathrm{d}x$ and

$$\begin{aligned} I_2 &= \int_0^2 \frac{x^2+2}{x+1}\mathrm{d}x = \int_1^3 \frac{(u-1)^2+2}{u}\mathrm{d}u \\ &= \int_1^3 \frac{u^2-2u+3}{u}\mathrm{d}u \\ &= \int_1^3 \left(u - 2 + \frac{3}{u}\right)\mathrm{d}u \\ &= \left[\frac{u^2}{2} - 2u + \ln u\right]_1^3 \\ &= 3\ln 3. \end{aligned}$$

Alternatively: Division of polynomials gives

$$x^2 + 2 = (x-1)(x+1) + 3.$$

Therefore

$$\begin{aligned} I_2 &= \int_0^2 \frac{x^2+2}{x+1}\mathrm{d}x \\ &= \int_0^2 \left(x - 1 + \frac{3}{x+1}\right)\mathrm{d}x \\ &= \left[\frac{x^2}{2} - x + 3\ln|x+1|\right]_0^2 \\ &= 3\ln 3. \end{aligned}$$

For the third integral: the function $f(x) = x^4 \sin x$ is odd, therefore

$$I_3 = \int_{-\pi}^{\pi} x^4 \sin x \,\mathrm{d}x = 0.$$

(One could also calculate an antiderivative of $f$ using integration by parts, but this is quite more tedious.)

**Oppgave 3** Consider the function

$$g : (0, +\infty) \to \mathbb{R}, \qquad g(x) = \frac{\ln x}{x} \cdot$$

Find all asymptotes of the graph of $g$. Make a rough sketch of the graph of $g$ which shows the intervals of monotonicity and convexity of $g$.

**Solution:** We have

$$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$$

so $x = 0$ is a vertical asymptote of the graph of $g$, and

$$\lim_{x \to +\infty} \frac{\ln x}{x} = 0$$

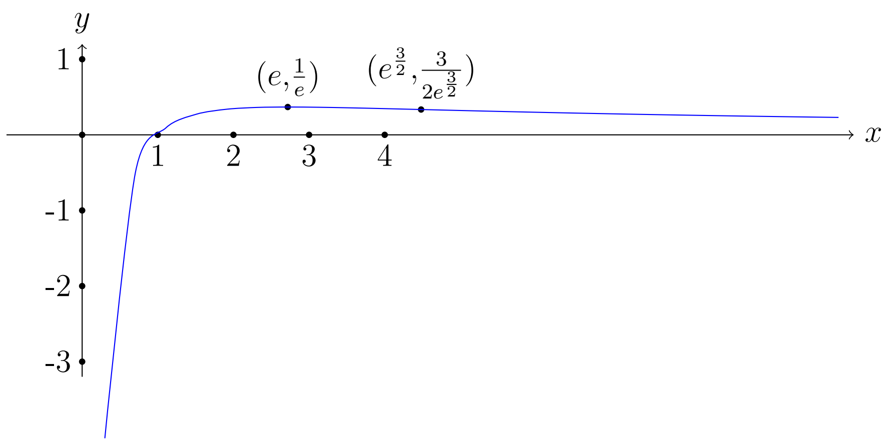
so $y = 0$ is a horizontal asymptote. Also

$$g'(x) = \frac{1 - \ln x}{x^2}$$

hence $g$ is strictly increasing on $(0, e]$ and strictly decreasing on $[e, +\infty)$. It has a local maximum at $x = e$ which is the number $g(e) = \frac{1}{e}$ and is also a global maximum.

$$g''(x) = \frac{2 \ln x - 3}{x^3}$$

and $g$ is concave on $(0, e^{\frac{3}{2}}]$ and convex on $[e^{\frac{3}{2}}, \infty)$.



**Oppgave 4** Solve the differential equation

$$(1+x^2)y' + xy = 0.$$

What is the order of this differential equation? Is it linear or not? If it is linear, is it homogeneous or inhomogeneous?

**Solution:** The differential equation is equivalent to

$$y' + \frac{x}{1+x^2}y = 0.$$

We need to find an antiderivative of $\frac{x}{1+x^2}$.

$$\int \frac{x}{1+x^2}\mathrm{d}x = \int \frac{1}{2}\frac{(1+x^2)'}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + c = \ln\sqrt{x^2+1} + c.$$

We multiply both sides by $e^{\ln\sqrt{x^2+1}} = \sqrt{x^2+1}$ and obtain

$$\begin{aligned} \sqrt{x^2+1}y' + \frac{x}{\sqrt{x^2+1}}y &= 0 \Leftrightarrow \\ \left(\sqrt{x^2+1}y\right)' &= 0 \Leftrightarrow \\ \sqrt{x^2+1}y &= c \Leftrightarrow \\ y = \frac{c}{\sqrt{x^2+1}}, & \qquad \text{with } c \in \mathbb{R} \text{ constant.} \end{aligned}$$

This is a first order linear homogeneous differential equation.

**Oppgave 5**

**a)** Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n}\,.$$

**b)** Is it true that whenever $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges? Justify your answer.

**Solution: a)** The corresponding sequence of partial sums is

$$S_N = \sum_{n=1}^{N} \frac{1}{n^2+3n}, \qquad N \geq 1.$$

We have

$$\frac{1}{n^2+3n} = \frac{1}{n(n+3)} = \frac{1}{3}\frac{(n+3)-n}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

hence

$$\begin{aligned} S_N &= \frac{1}{3}\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+3}\right) \\ &= \frac{1}{3}\left(1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \ldots + \frac{1}{N} - \frac{1}{N+3}\right) \\ &= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3}\right) \\ &\to \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18} \quad \text{as } N \to \infty, \end{aligned}$$

and therefore the series converges to the number $\frac{11}{18}$.

**b)** In general it is not true that whenever $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges. Take for example

$$a_n = \frac{1}{n}, \qquad n \geq 1.$$

Then $\lim_{n\to\infty} a_n = 0$, but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

**Oppgave 6** Does the integral

$$\int_0^\infty \frac{x\cos^2 x}{x^3+1}\mathrm{d}x$$

converge or diverge? Justify your answer.

**Solution:** For all $x \geq 1$ we have

$$\frac{x\cos x}{x^3+1} \leq \frac{x}{x^3+1} \leq \frac{1}{x^2}$$

and we know that the improper integral

$$\int_1^\infty \frac{1}{x^2}\mathrm{d}x$$

converges. Therefore

$$\int_1^\infty \frac{x\cos x}{x^3+1}\mathrm{d}x$$

converges, and so does

$$\int_0^\infty \frac{x\cos x}{x^3+1}\mathrm{d}x\,.$$

**Oppgave 7** Find the limit

$$\lim_{n\to\infty}\sum_{k=1}^n \frac{1}{2n+k}\,.$$

**Solution:** Let

$$f(x) = \frac{1}{2+x}, \qquad x \geq 0.$$

Then

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^n \frac{1}{2n+k} &= \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{n(2+\frac{k}{n})} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \frac{1}{2+\frac{k}{n}} \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x)\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{d}x}{2+x} = [\ln(2+x)]_0^1 = \ln\frac{3}{2}.
\end{aligned}$$

**Oppgave 8** Let

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} x, & \text{if } x < 0 \\ 1 + x^2, & \text{if } x \geq 0. \end{cases}$$

**a)** Use an $\varepsilon$-$\delta$ argument to show that $f$ is continuous at 1.

**b)** Does there exist a function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$? Justify your answer.

**Solution: a)** $f(1) = 2$ and $|f(x) - f(1)| = |x^2 - 1| = |x + 1||x - 1|$.
Let $\varepsilon > 0$. We have to prove that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|x - 1| < \delta \Rightarrow |f(x) - f(1)| < \varepsilon.$$

Whenever $|x - 1| < 1$ then $|x + 1| < 3$, and thus $|f(x) - f(1)| < 3|x - 1|$ . If we set

$$\delta = \min\left\{1, \frac{\varepsilon}{3}\right\} > 0$$

then $|x - 1| < \delta$ implies that

$$|f(x) - f(1)| < 3|x - 1| < 3\frac{\varepsilon}{3} = \varepsilon.$$

**b)** Assume such a function $F$ exists. Then $f = F'$ will have the Intermediate Value Property by Darboux's theorem, which is a contradiction.
Alternative answer: If $F'(x) = f(x)$ for all $x \in \mathbb{R}$, then

$$F(x) = \begin{cases} \frac{1}{2}x^2 + c_1, & \text{if } x < 0 \\ x + \frac{1}{3}x^3 + c_2, & \text{if } x > 0. \end{cases}$$

Since $F$ is differentiable at 0, it is also continuous at 0 and this implies that $c_1 = c_2$. Therefore

$$F(x) = \begin{cases} \frac{1}{2}x^2 + c & \text{if } x < 0 \\ x + \frac{1}{3}x^3 + c, & \text{if } x > 0. \end{cases}$$

Since $F'(0) = f(0) = 1$, we have

$$1 = \lim_{x \to 0^-} \frac{F(x) - F(0)}{x - 0} = \lim_{x \to 0^-} \frac{x^2}{2x} = 0,$$

which is a contradiction.

**Oppgave 9**

**a)** Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers which converges to 0. Prove that the sequence $(a_n)_{n=1}^{\infty}$ is bounded.

**b)** Give an example of a sequence of real numbers which is bounded but does not converge.

**Solution: a)** Let $\varepsilon = 1$. Since $\lim_{n\to\infty} a_n = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| < 1, \qquad \text{for all } n > n_0.$$

Set

$$M = \max\{1, |a_1|, \ldots, |a_{n_0}|\}.$$

Then $|a_n| \leq M$ for all $n \geq 1$, so the sequence $(a_n)_{n=1}^{\infty}$ is bounded.

**b)** Let

$$a_n = (-1)^n, \qquad n = 1, 2, \ldots$$

The sequence $(a_n)_{n=1}^{\infty}$ is bounded but does not converge.

**Oppgave 10** Let

$$g(x) = \begin{cases} \frac{1}{x^2}\sin^2\left(\frac{\pi x}{2}\right), & \text{if } x \neq 0 \\ \frac{\pi^2}{4}, & \text{if } x = 0 \end{cases} \qquad \text{and} \qquad F(x) = \int_1^x g(u)\mathrm{d}u\,.$$

**a)** Find the (maximal) domain of definition of $F$. Justify your answer.

**b)** Prove that $F$ is uniformly continuous on its domain of definition.

**c)** Prove that $F$ is invertible. Find the equation of the tangent line of the graph of $F^{-1}$ at the point $(0, F^{-1}(0))$.

**Solution: a)** The function $g$ is continuous at all points $x \neq 0$. We have

$$\lim_{x\to 0}\frac{1}{x^2}\sin^2\left(\frac{\pi x}{2}\right) = \frac{\pi^2}{4} = g(0),$$

therefore $g$ is continuous at all points in $\mathbb{R}$. Since $g$ is continuous, it is also Riemann integrable and the domain of definition of $F$ is equal to $\mathbb{R}$.

**b)** Since $g$ is continuous, by the Fundamental Theorem of Calculus the function $F$ is differentiable with

$$F'(x) = g(x) \qquad \text{for all } x \in \mathbb{R}.$$

Using the inequality $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$ we deduce that

$$|F'(x)| = |g(x)| \leq \frac{\pi^2}{4}.$$

That is, the derivative of $F$ is bounded, hence $F$ is uniformly continuous in $\mathbb{R}$.

**c)** Since $F'(x) = g(x)$, we have

$$F'(x) \geq 0 \quad \text{for all } x \in \mathbb{R}$$

and equality is true only when $x = 2n, n = \pm 1, \pm 2, \ldots$, that is, only on isolated points of $\mathbb{R}$, hence the function $F$ is strictly increasing and therefore also invertible. We have $F(1) = 0$ and thus

$$F^{-1}(0) = 1.$$

Also

$$(F^{-1})'(0) = \frac{1}{F'(F^{-1}(0))} = \frac{1}{F'(1)} = \frac{1}{g(1)} = 1,$$

so the equation of the tangent line of the graph of $F^{-1}$ at $(0, F^{-1}(0))$ is

$$y - F^{-1}(0) = (F^{-1})'(0)(x - 0) \quad \Rightarrow \quad y = x + 1.$$