NTNU
Kunnskap for en bedre verden

Institutt for matematiske fag

Eksamensoppgave i **MA1101 Grunnkurs i Analyse I**

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Eksamensdato: 2021

Eksamenstid (fra–til): :00 - :00

Hjelpemiddelkode/Tillatte hjelpemidler: B: Alle trykte og håndskrevne hjelpemidler tillatt. Bestemt, enkel kalkulator tillatt.

Målform/språk: bokmål

Antall sider: 7

Antall sider vedlegg: 2

Kontrollert av:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☒	

Dato Sign

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Oppgave 1 Let

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = xe^{-x} + 1.$$

a) Find the second degree Taylor polynomial of f at the point $x_0 = \ln 2$.

b) What are the global minimum and the global maximum of f?

Solution. a) We have

$$f'(x) = (1-x)e^{-x} \quad \text{and } f''(x) = (x-2)e^{-x}.$$

The second degree Taylor polynomial of f at x_0 is

$$\begin{aligned} P(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \\ &= 1 + \frac{\ln 2}{2} + \frac{1 - \ln 2}{2}(x - \ln 2) - \frac{2 - \ln 2}{4}(x - \ln 2)^2. \end{aligned}$$

b) By the first derivative test, we see that f is increasing on $(-\infty, 1]$ and decreasing on the set $[1, +\infty)$. Therefore f has a global maximum at the point 1, equal to $f(1) = 1 + \frac{1}{e}$. It does not have any global minimum because $\lim\limits_{x \to -\infty} f(x) = -\infty$.

Oppgave 2 Decide if each of the following statements is true or false. (You do not need to justify your answers).

a) Every Riemann–integrable function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$.

b) Every convergent sequence $(a_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ is also bounded.

c) Every continuous function $f : [0, 1] \to \mathbb{R}$ is also uniformly continuous.

d) $\lim\limits_{x \to 0} \dfrac{\sin x}{\sinh x} = 1.$

e) If $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable on $x_0 \in \mathbb{R}$ and $f''(x_0) = 0$, then x_0 is an inflection point (saddle point) of f.

Solution. a) False **b)** True **c)** True **d)** True **e)** False

Oppgave 3 Evaluate the following integrals.

$$\int_2^3 \frac{1}{x(x-1)^2}\mathrm{d}x, \quad \int_0^\pi x \sin x \,\mathrm{d}x, \quad \int_0^1 \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x.$$

Solution. We call the three integrals I_1, I_2, I_3 respectively. For the first of them, we search real numbers $A, B, C \in \mathbb{R}$ such that

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}, \qquad \text{for all } x \neq 0, 1.$$

This implies that

$$A(x-1)^2 + Bx(x-1) + Cx = 1 \quad \text{for all } x \neq 0, 1$$

and the same inequality will also be true for all $x \in \mathbb{R}$ (because we have two polynomials equal at infinitely many real numbers). Setting $x = 0$ we get $A = 1$. Setting $x = 1$ we get $C = 1$. Finally, for $x = 2$ we deduce that $B = -1$. Thus

$$\frac{1}{x(x-1)^2} = \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}$$

and

$$\begin{aligned} I_1 &= \int_2^3 \Big(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\Big)\mathrm{d}x \\ &= \Big[\ln|x| - \ln|x-1| - \frac{1}{x-1}\Big]_2^3 \\ &= \ln\frac{3}{2} - \ln 2 - \Big(\frac{1}{2} - 1\Big) = \frac{1}{2} + \ln\frac{3}{4}. \end{aligned}$$

For the second integral, we use integration by parts to deduce

$$\begin{aligned} I_2 &= \int_0^\pi x(-\cos x)'\mathrm{d}x = [-x\cos x]_0^\pi + \int_0^\pi \cos x \mathrm{d}x \\ &= \pi + [\sin x]_0^\pi = \pi. \end{aligned}$$

For I_3 we make the substitution $x = \sinh u$. The new differential is $\mathrm{d}x = \cosh u \mathrm{d}u$, and the new limits of integration are

$$u_1 = 0 \quad \text{and} \quad u_2 = \mathrm{arcsinh} 1 = \ln(1+\sqrt{2}).$$

Thus

$$I_3 = \int_0^{\ln(1+\sqrt{2})} \frac{\cosh u \mathrm{d}u}{\sqrt{1+\sinh^2 u}} = \int_0^{\ln(1+\sqrt{2})} 1 \mathrm{d}u = \ln(1+\sqrt{2}).$$

Oppgave 4 Solve the integral equation

$$y(x) = \int_0^x 2t \cdot (1 + y^2(t))\, \mathrm{d}t.$$

Solution. Differentiation yields

$$\begin{aligned} y' = 2x(1+y^2) &\Rightarrow \frac{\mathrm{d}y}{1+y^2} = 2x\mathrm{d}x \\ &\Rightarrow \int \frac{\mathrm{d}y}{1+y^2} = \int 2x\mathrm{d}x \\ &\Rightarrow \arctan y = x^2 + C, \qquad \text{where } C \in \mathbb{R}. \end{aligned}$$

We now find the value of the constant C. When $x = 0$, we have $y(0) = 0$, thus

$$\arctan 0 = 0 + C \Rightarrow C = 0.$$

Therefore $\arctan y = x^2$, whence

$$y(x) = \tan x^2, \qquad x \in \mathbb{R}.$$

Oppgave 5 Evaluate the following series.

a)

$$\sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)}\ .$$

b)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Solution. a) We note that

$$\frac{1}{(n-1)(n+1)} = \frac{1}{2} \cdot \frac{2}{(n-1)(n+1)} = \frac{1}{2} \cdot \frac{(n+1)-(n-1)}{(n+1)(n-1)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

Therefore for N large enough, the N–th partial sum of the series is

$$\begin{aligned} S_N &= \sum_{n=2}^{N} \frac{1}{(n+1)(n-1)} = \frac{1}{2} \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \ldots + \frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right). \end{aligned}$$

Therefore $\lim_{N\to\infty} S_N = \frac{3}{4}$ and the series converges to the number $\frac{3}{4}$.

b) We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x \in \mathbb{R}.$$

Setting $x = 2$ we get

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{2^n}{n!},$$

so the series converges to the number $e^2 - 1$.

Oppgave 6 Does the integral

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^4+3} \mathrm{d}x$$

converge or diverge? Justify your answer.

Solution. For any $x \in \mathbb{R}$ we have

$$\frac{x^2}{x^4+3} \leqslant \frac{1}{x^2}$$

and thus the improper integrals

$$\int_{-\infty}^{-1} \frac{x^2}{x^4+3} \mathrm{d}x, \qquad \int_{1}^{+\infty} \frac{x^2}{x^4+3} \mathrm{d}x$$

both converge. Hence so do the improper integrals

$$\int_{-\infty}^{0} \frac{x^2}{x^4+3} \mathrm{d}x, \qquad \int_{0}^{+\infty} \frac{x^2}{x^4+3} \mathrm{d}x$$

and also the given integral.

Oppgave 7 Find the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n(k+n)}.$$

Solution. We have

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n(k+n)} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}}{\frac{k}{n}+1} \\ &= \int_0^1 \frac{x}{x+1} \mathrm{d}x \\ &= \int_0^1 \left(1 - \frac{1}{x+1}\right) \mathrm{d}x \\ &= \Big[x - \ln|1+x|\Big]_0^1 \\ &= 1 - \ln 2.\end{aligned}$$

Oppgave 8 Let

$$A = [-1, 0) \cup (1, 2).$$

a) Find $\sup A$. Justify your answer.

b) Find (without proof) $\inf A$, $\max A$ and $\min A$.

Solution. a) We have $\sup A = 2$. Indeed: if $x \in A$, then either $x \in [-1, 0)$ or $x \in (1, 2)$. In both cases, $x < 2$. Hence 2 is an upper bound of the set A. Now let $\varepsilon > 0$. If $0 < \varepsilon < 1$, then the number

$$x_\varepsilon := 2 - \frac{\varepsilon}{2} \in A$$

and

$$2 - \varepsilon < x_\varepsilon < 2.$$

When $\varepsilon \geqslant 1$, then we set $x_\varepsilon = \dfrac{3}{2} \in A$, and thus

$$2 - \varepsilon < x_\varepsilon < 2.$$

b) We have $\inf A = \min A = -1$, while $\max A$ does not exist.

Oppgave 9 Suppose the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{n\to\infty} (a_{n-1}a_{n+1} - a_n^2) = 0.$$

Justify your answer. *(Anything that was taught in the lectures can be used without proof.)*

Solution. Because the series $\sum_{n=1}^{\infty} a_n$ converges, we have

$$\lim_{n\to\infty} a_n = 0.$$

Therefore also

$$\lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_{n+1} = 0$$

whence the requested follows.

Oppgave 10 Find the values of the real number $\lambda \in \mathbb{R}$ for which the integral

$$\int_0^{\infty} \frac{1}{(e^x - e^{-x})^{\lambda}} \mathrm{d}x$$

converges and diverges, respectively. Justify your answer.

Solution. The given integral converges if and only if the integrals

$$J_1 = \int_0^1 \frac{\mathrm{d}x}{(e^x - e^{-x})^{\lambda}} \qquad \text{and} \qquad J_2 = \int_1^{\infty} \frac{\mathrm{d}x}{(e^x - e^{-x})^{\lambda}}$$

both converge (either as improper integrals or as Riemann integrals). Regarding J_1, we observe that

$$\lim_{x\to 0^+} \frac{e^x - e^{-x}}{2x} = \lim_{x\to 0^+} \frac{e^x + e^{-x}}{2} = 1.$$

We therefore compare J_1 with the integral $\int_0^1 \frac{\mathrm{d}x}{(2x)^{\lambda}}$, for which we know that

$$\int_0^1 \frac{\mathrm{d}x}{(2x)^{\lambda}} \quad \begin{cases} = \infty, & \text{if } \lambda \geqslant 1 \\ < \infty, & \text{if } \lambda < 1. \end{cases}$$

Hence J_1 converges if and only if $\lambda < 1$. Regarding the integral J_2, we observe that for any $\lambda \leqslant 0$, we have

$$\frac{1}{(e^x + e^{-x})^\lambda} = (e^x + e^{-x})^{-\lambda} \geqslant 1 \qquad \text{for all } x \geqslant 1$$

and thus J_2 diverges. On the other hand, when $\lambda > 0$ we have

$$e^x - e^{-x} \geqslant \frac{e^x}{2} \quad \Rightarrow \quad \frac{1}{(e^x - e^{-x})^\lambda} \leqslant \left(\frac{2}{e^x}\right)^\lambda$$

for all $x \geqslant 1$, and therefore J_2 converges.

We conclude that the integral converges if and only if $0 < \lambda < 1$.

Dette er en vedleggsside.

Og dette er en til.