Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

Løsningsforslag — Øving

1 Hvilke av følgende utsagn er korrekte? Svar med «Sann» eller «Usann». *Begrunnelse trengs ikke på denne oppgaven.*

a) Hvis en kontinuerlig deriverbar funksjon f på $\mathbb{R}$ tilfredstiller at $f'(x) < 0$ for $x < 0$ og $f'(x) > 0$ for $x > 0$, må f ha et minimum ved $x = 0$.

b) Hver kontinuerlig og deriverbar funksjon på et åpent intervall (a, b) oppnår et maksimum.

c) Hvis $(a_n)_{n=1}^{\infty}$ konvergerer mot 0 når $n \to \infty$, så er rekken $\sum_{n=1}^{\infty} a_n$ konvergent.

d) Hvis f og g er kontinuerlige funksjoner på et intervall I, så er produktfunksjonen $h = fg$ også kontinuerlig på I.

e) Produktet av to diskontinuerlige funksjoner kan ikke være kontinuerlig.

f) Hvis en kontinuerlig funksjon f tilfredsstiller $0 < f(x) < \frac{1}{x^2}$ for alle $x \geq 1$, så konvergerer $\int_1^{\infty} f(x)\,dx$.

g) Hvis en kontinuerlig funksjon f tilfredsstiller $f(x) > \frac{1}{x}$ for alle $x \geq 1$, så divergerer $\int_1^{\infty} f(x)\,dx$.

h) Hvis en funksjon $f\colon \mathbb{R} \to \mathbb{R}$ tilfredstiller $|f(x) - f(y)| \leq L|x - y|$ for en bestemt konstant $L > 0$ og alle $x, y \in \mathbb{R}$, så er f kontinuerlig.

i) Alle deriverbare funksjoner $f\colon \mathbb{R} \to \mathbb{R}$ som tilfredstiller $f(x) = f'(x)$, $x \in \mathbb{R}$, kan skrives $f(x) = Ce^x$ for en konstant $C \in \mathbb{R}$.

j) For ethvert $a \in \mathbb{R}$, eksisterer en funksjon f slik at $f(x) = -f(-x)$ og $\int_{-a}^{2a} f(x)\,dx = a$.

Solution:
a) True: Decreasing then increasing, hence minimum.
b) False: E.g. $(a, b) = (0, 1), f(x) = x$.
c) False: E.g. $a_n = 1/n$.
d) True: It is true.
e) False: E.g. $f(x) = g(x) = -1$ for $x \leq 0$ and $f(x) = g(x) = 1$ for $x > 0$.
f) True: Comparison.
g) True: Comparison.
h) True: $(f(x + h) - f(x)) \to 0$.
i) True: Property of e^x.
i) True: $\int_{-a}^{a} f(x)dx = 0$, then extend.

2 La

$$f(x) = xe^{1/x}, x \neq 0.$$

Bestem intervallene der f er voksende eller avtakende. Bestem også alle vertikale, horisontale, og skrå asymptoter til f. Lag en skisse av grafen til $y = f(x)$ med hjelp av dine svar.

Solution: We compute the derivative of $f(x)$ as

$$f'(x) = e^{1/x} + x\frac{-1}{x^2}e^{1/x} = e^{1/x}[1 - \frac{1}{x}].$$

The $e^{1/x}$ factor is positive for all x so the sign of f' is determined by $1 - \frac{1}{x}$ so we conclude that f is decreasing on $(0, 1)$ and increasing on $(-\infty, 0) \cup (1, \infty)$.

For vertical asymptotes we only have $x = 0$ where $e^{1/x}$ explodes. There are no horizontal asymptotes as $f(x) \to \pm\infty$ as $x \to \infty$. More specifically regarding the behavior at infinity, for large x, $e^{1/x} \approx 1$ and so f will behave as $g(x) = x + b$ for some b. To find the value of b, we compute the limit

$$\lim_{|x|\to\infty} f(x) - g(x) = \lim_{|x|\to\infty} x[e^{1/x} - 1] = \lim_{|x|\to\infty} \frac{e^{1/x} - 1}{\frac{1}{x}} = \lim_{|x|\to\infty} \frac{\frac{-1}{x^2}e^{1/x}}{\frac{-1}{x^2}} = 1$$

and so the linear asymptote is $g(x) = x + 1$ for both $x \to +\infty, -\infty$.

To sketch the graph, we first evaluate $f(1) = e^{\frac{1}{1}} \approx 2.7$. To the left of $x = 0$, f is approximately flat as can be seen by computing $\lim_{x\to 0^-} f'(x)$. Putting this together, we get the following sketch with the blue line indicating the linear asymptote.

3 **a)** Beregn Taylorpolynomet av grad 3 rundt punktet $x = 0$ til funksjonen

$$f(x) = \int_0^x \cos(\sin(t))\, dt.$$

b) Finn en tilnærming til $f(0,1)$ med feil mindre enn $0,0005$.

Hint: Husk at

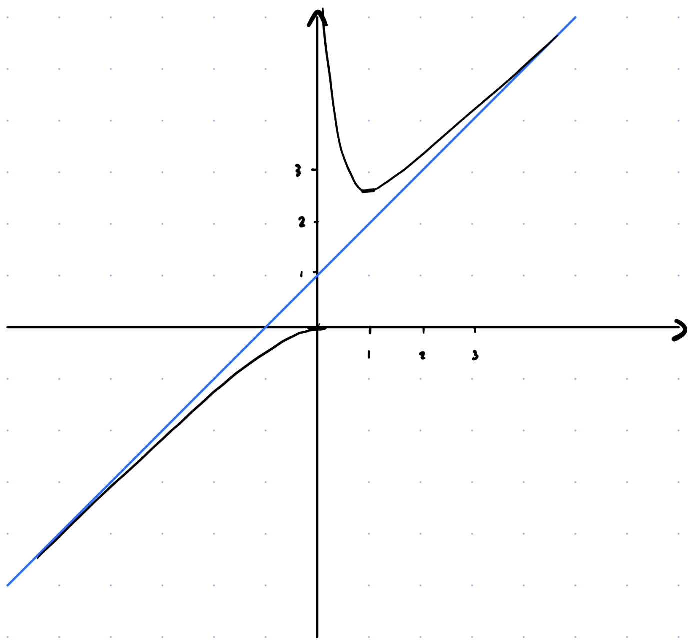
$$f(x) = T_n(x) + R_n(x), \qquad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Solution: a) We compute f', f'', f''' as follows:

$$\begin{aligned}
f'(x) &= \cos(\sin(x)) \implies f'(0) = 1,\\
f''(x) &= -\cos(x)\sin(\sin(x)) \implies f''(0) = 0\\
f'''(x) &= \sin(x)\sin(\sin(x)) - \cos^2(x)\cos(\sin(x)) \implies f'''(0) = -1.
\end{aligned}$$

Hence the third order Taylor polynomial of f around $x = 0$ is

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x - \frac{1}{6}x^3.$$



Figur 1: Sketch of $f(x)$

b) Recall that the error of using the first n terms of the Taylor polynomial to approximate the function value is bounded by

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}x^{n+1}.$$

If we attempt to use the Taylor polynomial we calculated in **a)**, we set $n = 2$ and bound the value of $R_2(0.1)$ as

$$|R_2(0.1)| = \frac{|\sin(c)\sin(\sin(c)) - \cos^2(c)\cos(\sin(c))|}{3!}0.1^3 \leq \frac{2}{3!}0.1^3 = \frac{0.001}{3} \leq \frac{0.001}{2} = 0.0005.$$

Hence an appropriate approximation for $f(0.1)$ is

$$f(0.1) \approx 0.1$$

which has error smaller than 0.0005.

4 **a)** Beregn $\lim_{n\to\infty} \frac{n^{2024} + 2n^9 + 4n^3 + n\sin n}{2n^{2024} + n^6 + 3n^2 + 2^{-n}}$

b) Beregn $\lim_{n\to\infty} (\sqrt{n^2+n} - \sqrt{n^2-2n})$.

c) Vis at

$$\lim_{n\to\infty} \frac{2^n}{\sqrt{n!}} = 0.$$

Hint: Sammenlign faktorer i telleren med faktorer i nevneren.

Solution:

a) By dividing by n^{2024} on both the numerator and denominator, we get

$$\lim_{n\to\infty} \frac{n^{2024} + 2n^9 + 4n^3 + n\sin n}{2n^{2024} + n^6 + 3n^2 + 2^{-n}} = \lim_{n\to\infty} \frac{1 + \underbrace{\frac{2n^9 + 4n^3 + n\sin n}{n^{2024}}}_{\to 0}}{2 + \underbrace{\frac{n^6 + 3n^2 + 2^{-n}}{n^{2024}}}_{\to 0}} = \frac{1}{2}.$$

b) We multiply and divide by the conjugate

$$\begin{aligned}\lim_{n\to\infty} \sqrt{n^2+n} - \sqrt{n^2-2n} &= \lim_{n\to\infty} \frac{n^2+n-n^2+2n}{\sqrt{n^2+n}+\sqrt{n^2-2n}} \\ &= \lim_{n\to\infty} \frac{3n}{\sqrt{n^2+n}+\sqrt{n^2-2n}} \\ &= \lim_{n\to\infty} \frac{3}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{2}{n}}} = \frac{3}{2}.\end{aligned}$$

c) To get rid of the square root, we square each term and show that that quantity approaches 0 instead.

$$\lim_{n\to\infty} \frac{2^n}{\sqrt{n!}} = \sqrt{\lim_{n\to\infty} \frac{4^n}{n!}}.$$

From here, we can apply the same argument as in the original exam. Indeed,

$$\frac{4^n}{n!} = \frac{4}{1}\cdot\frac{4}{2}\cdot\frac{4}{3}\cdots\frac{4}{n-1}\cdot\frac{4}{n} \le \frac{4}{1}\cdot\frac{4}{2}\cdot\frac{4}{3}\cdot\frac{4}{4}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdots\frac{4}{5}$$

by a trivial esimate. Hence $\frac{4^n}{n!} \le \frac{4^4}{1\cdot2\cdot3\cdot4}\left(\frac{4}{5}\right)^{n-4}$ which clearly goes to zero as $n\to\infty$.

5 Betrakt følgen $(a_n)_{n=1}^{\infty}$, rekursivt definiert av at $a_1 = 0$, $a_2 = 3$, og at

$$a_{n+1} = \frac{a_n + a_{n-1}}{2}, \qquad n \ge 2.$$

a) Forklar med ord hvordan følgen oppfører seg. Illustrer gjennom å tegne noen verdier av a_n.

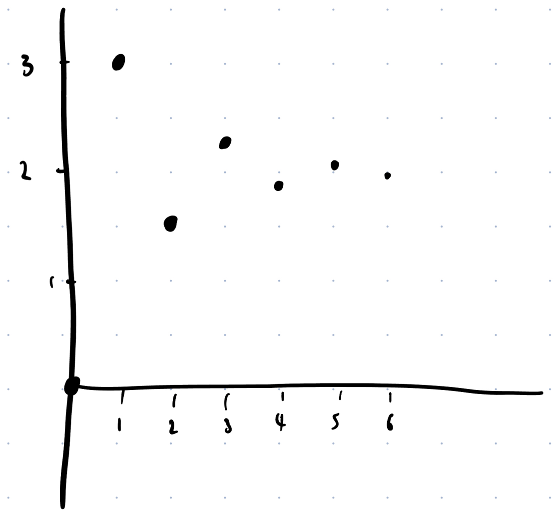
b) Vis ved hjelp av induksjon at

$$a_{n+1} = \sum_{j=1}^{n} (-1)^{j-1}\frac{3}{2^{j-1}}, \qquad n \ge 2.$$

Hint: I induksjonssteget kan det være til hjelp å bruke at

$$(-1)^{k-1}\frac{3}{2^k} = (-1)^{k-1}\frac{3}{2^{k-1}} + (-1)^k\frac{3}{2^k}.$$

c) Bruk formelen ovenfor for å vise at $\lim_{n\to\infty} a_n$ eksisterer.



Figur 2: Sketch of the sequence

Solution:
a) The solution tends towards the mean of the last two values, this means that it should oscillate with smaller and smaller amplitudes until eventually becoming stable.

b) For $n = 2$, we have

$$a_3 = (-1)^0 \frac{3}{2^0} + (-1)^1 \frac{3}{2^1} = 3 - \frac{3}{2} = \frac{3}{2}$$

which is correct. For the induction step, assume the statement is true for some $n = k$, i.e.

$$a_{k+1} = \sum_{j=1}^{k} (-1)^{j-1} \frac{3}{2^{j-1}}$$

We need to prove it for $n = k + 1$ and so write

$$\begin{aligned} a_{k+2} &= \frac{a_{k+1} + a_k}{2} \\ &= \frac{1}{2} \left(\sum_{j=1}^{k} (-1)^{j-1} \frac{3}{2^{j-1}} + \sum_{j=1}^{k-1} (-1)^{j-1} \frac{3}{2^{j-1}} \right) \\ &= \frac{1}{2} \left(2 \sum_{j=1}^{k-1} (-1)^{j-1} \frac{3}{2^{j-1}} + (-1)^{k-1} \frac{3}{2^{k-1}} \right) \\ &= \sum_{j=1}^{k-1} (-1)^{j-1} \frac{3}{2^{j}} + (-1)^{k-1} \frac{3}{2^{k}}. \end{aligned}$$

In order for this to match the full sum up to $k + 1$, we need to show that the extra term $(-1)^{k-1} \frac{3}{2^k}$ is equal to the two last terms in the full sum $a_{k+2} = \sum_{j=1}^{k+1} (-1)^{j-1} \frac{3}{2^{j-1}}$. Indeed,

$$\underbrace{(-1)^{k-1} \frac{3}{2^{k-1}}}_{j=k} + \underbrace{(-1)^{k} \frac{3}{2^{k}}}_{j=k+1} = (-1)^k \frac{3}{2^k} \Big[-2 + 1 \Big] = (-1)^{k-1} \frac{3}{2^k}$$

and so we are done.
c) Using the sum, we find

$$\lim_{n\to\infty} a_n = \sum_{j=1}^{\infty} 3\left(\frac{-1}{2}\right)^{j-1} = 3\sum_{j=0}^{\infty}\left(\frac{-1}{2}\right)^{j} = \frac{3}{1-\frac{-1}{2}} = \frac{3}{\frac{3}{2}} = 2.$$

6 **a)** Beregn $\int_3^5 xe^x\,dx$.

b) Beregn $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$.

c) Beregn $\int_0^\pi \cos^2 x \sin^3 x\,dx$.

Solution:

a) To evaluate $\int_3^5 xe^x\,dx$, we use integration by parts with $u = x, v' = e^x$, then $v = e^x$ and

$$\int_3^5 xe^x\,dx = \Big[xe^x\Big]_3^5 - \int_3^5 e^x\,dx = 5e^5 - 3e^3 - e^5 + e^3 = 4e^5 - 2e^3.$$

b) To evaluate $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$, we can use the substitution $u = \ln x$.

$$du = \frac{1}{x}dx$$
$$x = e^u$$

When $x = 2, u = \ln(2)$ and when $x = \infty$, $u = \infty$ so the integral becomes

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \int_{\ln(2)}^\infty \frac{1}{u^2}\,du = \frac{1}{\ln(2)}.$$

c) We want to use a substitution of the form $u = \cos(x)$ and then divide away the $\sin(x)$ factor we get in $dx = \frac{-1}{\sin(x)}$. To turn $\sin(x)^3$ into something with $\sin(x)$ alone and $\cos(x)$, we use $\sin(x)^2 = 1 - \cos(x)^2$ so the integral becomes

$$\int_0^\pi \cos^2(x)\sin^3(x)\,dx = \int_0^\pi \cos^2 x \sin x(1-\cos^2 x)\,dx.$$

Now making the substitution, we get

$$\begin{aligned}\int_0^\pi \cos^2 x \sin x(1-\cos^2 x)\,dx &= \int_{x=0}^{x=\pi} u^2 \sin x(1-u^2)\frac{-1}{\sin(x)}\,du \\ &= \int_{x=0}^{x=\pi} u^4 - u^2\,du \\ &= \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_{x=0}^{x=\pi}.\end{aligned}$$

Recall that $\cos(0) = 1$ and $\cos(\pi) = -1$, hence we can evaluate this as

$$\begin{aligned}\int_0^\pi \cos^2(x)\sin^3(x)\,dx &= \frac{-1}{5} - \frac{-1}{3} - \left(\frac{1}{5} - \frac{1}{3}\right)\\ &= \frac{-2}{5} + \frac{2}{3} = \frac{4}{15}.\end{aligned}$$

7 La $f\colon [0,\infty) \to (0,\infty)$ være en positiv, kontinuerlig funksjon, slik at $f(x) \leq 1$ for $x \in [0,1]$. Anta videre at

$$f(x+1) \leq \frac{f(x)}{2}, \qquad x \geq 0.$$

Vis at

$$\sup_{n\geq 1} \int_0^n f(x)\,dx < \infty,$$

og dermed at det uegentlige integralet

$$\int_0^\infty f(x)\,dx$$

konvergerer.
Hint: Hvor stor kan f være på $[1,2]$, $[2,3]$, $[3,4]$,...?

Solution:
We will bound f on each of the intervals $[0,1], [1,2], [2,3], \ldots$ and use that

$$\int_0^\infty f(x)\,dx \leq \sum_{n=0}^\infty \int_n^{n+1} f(x)\,dx \leq \sum_{n=0}^\infty \sup_{x\in[n,n+1]} |f(x)|.$$

From the assumption, we have that f more than halves when we move to the right by 1. If we let C denote the bound for f on $[0,1]$ (which is finite since f is continuous and $[0,1]$ is compact), we have that

$$\sup_{x\in[1,2]} f(x) = \sup_{x\in[0,1]} f(x+1) \leq \sup_{x\in[0,1]} \frac{f(x)}{2} \leq \frac{C}{2}.$$

Similarly we have that

$$\sup_{x\in[2,3]} f(x) = \sup_{x\in[1,2]} f(x+1) \leq \sup_{x\in[1,2]} \frac{f(x)}{2} \leq \frac{C}{2^2}$$

and in general,

$$\sup_{x\in[n,n+1]} f(x) \leq \frac{C}{2^n}.$$

Plugging this into the sum bounding our integral, we get

$$\int_0^\infty f(x)\,dx \leq \sum_{n=0}^\infty \sup_{x\in[n,n+1]} f(x) \leq C\sum_{n=0}^\infty \frac{1}{2^n} = 2C < \infty.$$