Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1101 Grunnkurs i analyse I Høst 2023

Løsningsforslag — Eksamen Høst 2023

1 Hvilke av følgende utsagn er korrekte? Svar med «Sann» eller «Usann». *Begrunnelse trengs ikke på denne oppgaven.*

a) Hvis en deriverbar funksjon $f : [0,1] \to [0,1]$ har et maksimum i $x = 1/2$, så er $f'(1/2) = 0$.

b) Hvis $(a_n)_{n=1}^{\infty}$ er en konvergent følge, kan ikke følgen $(1/a_n)_{n=1}^{\infty}$ konvergere.

c) Funksjonen $f : (0,1] \to \mathbb{R}$ definert ved $f(x) = x\sin(1/x)$ kan utvides til en kontinuerlig funksjon på $[0,1]$.

d) Hvis grensen $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ eksisterer, er f kontinuerlig i x.

e) Enhver kontinuerlig funksjon på $\mathbb{R}$ er begrenset.

f) Hvis en kontinuerlig funksjon f tilfredsstiller $0 < f(x) < \frac{1}{x}$ for alle $x \geq 1$, så konvergerer $\int_1^{\infty} f(x)\,dx$.

g) Hvis en kontinuerlig funksjon f tilfredsstiller $f(x) > \frac{1}{x^2}$ for alle $x \geq 1$, så divergerer $\int_1^{\infty} f(x)\,dx$.

h) Hvis $(a_n)_{n=1}^{\infty}$ er en konvergent følge, så er $\lim_{n\to\infty} |a_n - a_{2n}| = 0$.

i) Hvis $f : (0,1) \to \mathbb{R}$ er kontinuerlig på $(\varepsilon, 1)$ for hver $0 < \varepsilon < 1$, så er den kontinuerlig på hele $(0,1)$.

j) Det finnes reelle tall a, b, c slik at

$$\int_{-\pi}^{\pi} \sin(x)\cos(x)[ax^4 + bx^2 + c]\,dx = 2\pi.$$

Solution:
a) True: Critical point derivative zero.
b) False: E.g. $a_n = 1$.
c) True: $f(x) \to 0$ as $x \to 0$.
d) True: Differentiable implies continuous.
e) False: E.g. $f(x) = x$.
f) False: E.g. $f(x) = \frac{1}{2x}$.
g) False: E.g. $f(x) = \frac{2}{x^2}$.
h) True: Linearity of limits.
i) True: For any $x \in (0,1)$, $x \in (\varepsilon, 1)$ for some $\varepsilon > 0$. Continuity is a pointwise property.
j) False: Odd function.

2 La

$$f(x) = \frac{\ln x}{x}, \qquad x > 0.$$

Bestem intervallene der f er voksende eller avtakende. Bestem også alle vertikale, horisontale, og skrå asymptoter til f. Lag en skisse av grafen til $y = f(x)$ med hjelp av dine svar. Finn det største verdiet som f antar.

Solution:

To find where $f(x)$ is increasing or decreasing, we first find its derivative:

$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$$

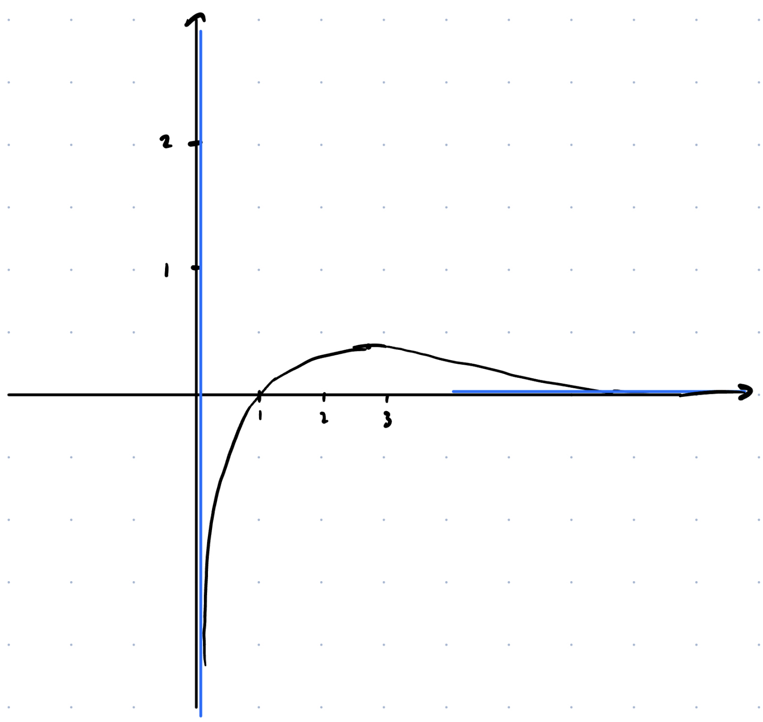
From this we see that f is increasing on $(0, e)$ and decreasing on (e, ∞) while $x = 1$ is a critical point.

As $x \to 0$, $f(x) \to -\infty$ so $x = 0$ is a vertical asymptote. Meanwhile as $x \to \infty$, $f(x) \to 0$ so $y = 0$ is a horizontal asymptote.

Since $f(x)$ increases for $x < e$ and decreases for $x > e$, we have a maximum at $x = e$. The maximum value is

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}.$$

The function takes on the value 0 at $x = 1$ which we include in the graph.



Figur 1: Sketch of $f(x)$

3 **a)** Beregn Taylorpolynomet av grad 3 rundt punktet $x = 0$ til funksjonen

$$f(x) = xe^{-x}.$$

b) Ved å bruke Taylorpolynomet av grad 2 rundt punktet $x = 0$, finn en tilnærming til $f(0,1)$ med feil mindre enn $0,0005$.

Hint: Husk at

$$f(x) = T_n(x) + R_n(x), \qquad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Solution: a) We compute f', f'', f''' as follows:

$$\begin{aligned} f'(x) &= (1-x)e^{-x} \implies f'(0) = 1, \\ f''(x) &= (x-2)e^{-x} \implies f''(0) = -2 \\ f'''(x) &= (3-x)e^{-x} \implies f'''(0) = 3. \end{aligned}$$

Hence the third order Taylor polynomial of f around $x = 0$ is

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x - x^2 + \frac{1}{2}x^3.$$

b) Recall that the error of using the first n terms of the Taylor polynomial to approximate the function value is equal to

$$R_n(x) = \frac{f^{n+1}(c)}{(n+1)!}x^{n+1}.$$

If we attempt to use the Taylor polynomial we calculated in **a)**, we set $n = 2$ and bound the value of $R_2(0.1)$ as

$$|R_2(0.1)| = \frac{(3-c)e^{-c}}{3!}0.1^3 < \frac{3e^0}{3!}0.1^3 = 0.0005.$$

Hence an appropriate approximation for $f(0.1)$ is

$$f(0.1) \approx 0.1 - 0.1^2 = 0.09$$

which has error smaller than 0.0005.

4 **a)** Beregn $\lim_{n\to\infty} \frac{5n^{2023} + 3n^9 + 2n^3 + 8}{2n^{2023} + n^7 + 8n^2 + 2}$

b) Beregn $\lim_{n\to\infty} n \tan \frac{1}{n}$.

c) Vis at

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

Hint: Sammenlign hver faktor i telleren med en faktor i nevneren.

Solution:

a)
By dividing by n^{2023} on both the numerator and denominator, we get

$$\lim_{n\to\infty} \frac{5n^{2023} + 3n^9 + 2n^3 + 8}{2n^{2023} + n^7 + 8n^2 + 2} = \lim_{n\to\infty} \frac{5 + \underbrace{\frac{3n^9 + 2n^3 + 8}{n^{2023}}}_{\to 0}}{2 + \underbrace{\frac{n^7 + 8n^2 + 2}{n^{2023}}}_{\to 0}} = \frac{5}{2}.$$

b)
We rewrite the limit as

$$\lim_{n\to\infty} n \tan\frac{1}{n} = \lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} \cos\frac{1}{n} = 1$$

since both factors converge to 1 as $n \to \infty$.

c)
Both the numerator and denominator have n factors and so we can group them. Indeed,

$$\frac{2^n}{n!} = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdots \frac{2}{n-1} \cdot \frac{2}{n} \le \frac{2}{1} \cdot 1 \cdot 1 \cdot 1 \cdots 1 \cdot \frac{2}{n} = \frac{4}{n}.$$

Hence $0 \le \frac{2^n}{n!} \le \frac{4}{n}$, demonstrating that $\frac{2^n}{n!}$ goes to zero as $n \to \infty$, by the squeeze rule.

5 Betrakt følgen $(a_n)_{n=1}^{\infty}$, rekursivt definiert av at

$$a_1 = 0, \qquad a_{n+1} = \sqrt{6 + a_n}, \; n \ge 1.$$

a) Vis ved hjelp av induksjon at følgen er voksende.

b) Vis ved hjelp av induksjon at følgen er begrenset.

c) Motiver at følgen er konvergent og beregn $\lim_{n\to\infty} a_n$.

Solution:

a) Consider the function $f(x) = \sqrt{6 + x}$ which clearly is increasing. The show that the sequence is increasing, we first note that $a_2 = \sqrt{6} > a_1 = 0$. For the induction step,

$$a_n \le a_{n+1} \implies a_{n+1} = f(a_n) \le f(a_{n+1}) = a_{n+2}.$$

b) Here it helps to have a guess for what the "ceiling" for the sequence should be. It holds that $\sqrt{6 + x} = x$ when $x = 3$ and so it should be reasonable that this is the upper bound. Clearly $a_1, a_2 \le 3$ and so we prove by induction

$$a_n \le 3 \implies a_{n+1} = \sqrt{6 + a_n} \le \sqrt{6 + 3} = \sqrt{9} = 3.$$

c) The sequence converges because it is both bounded and increasing. To compute the limit L, we can use that $f(x)$ is a continuous function to say that

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n + 6} = \sqrt{6 + \lim_{n\to\infty} a_n} = \sqrt{6 + L} \implies L = 3.$$

6 **a)** Beregn $\displaystyle\int \frac{x}{\sqrt{3x-1}}\,dx$.

b) Beregn $\displaystyle\int_0^\infty \frac{4x}{x^2+3}\,dx$.

c) Beregn $\displaystyle\int_0^1 \frac{1}{(1+x^2)^{3/2}}\,dx$. *Hint:* $x = \tan(u)$.

Solution

a) To calculate $\displaystyle\int \frac{x}{\sqrt{3x-1}}\,dx$, let $u = 3x - 1$, so $du = 3dx$ or $dx = \frac{du}{3}$.

$$\begin{aligned}\int \frac{x}{\sqrt{3x-1}}\,dx &= \frac{1}{9}\int \frac{u+1}{\sqrt{u}}\,du \\ &= \frac{1}{3}\int u^{1/2}\,du + \frac{1}{9}\int u^{-1/2}\,du \\ &= \frac{2}{27}u^{3/2} + \frac{2}{3}u^{1/2} + C \\ &= \frac{2}{27}(3x-1)\sqrt{3x-1} + \frac{2}{9}\sqrt{3x-1} + C \\ &= \frac{2}{27}\sqrt{3x-1}(3x+2) + C.\end{aligned}$$

b) To calculate $\displaystyle\int_0^\infty \frac{4x}{x^2+3}\,dx$, we can make the substitution $u = x^2 + 3$, so $du = 2xdx$ or $dx = \frac{du}{2x}$.

$$\begin{aligned}\int_0^\infty \frac{4x}{x^2+3}\,dx &= 2\int_3^\infty \frac{1}{u}\,du \\ &= 2\left[\ln|u|\right]_3^\infty \\ &= 2(\ln\infty - \ln 3) \\ &= \infty.\end{aligned}$$

The integral diverges.

c) To calculate $\displaystyle\int_0^1 \frac{1}{(1+x^2)^{3/2}}\,dx$, we can make the substitution $x = \tan(u)$, so $dx = \frac{1}{\cos(u)^2}du$. When $x = 0$, $u = 0$ and when $x = 1$, $u = \pi/4$. Using that $1 + \tan(u)^2 = \frac{1}{\cos(u)^2}$

we get

$$\int_0^1 \frac{1}{(1+x^2)^{3/2}}\,dx = \int_0^{\pi/4} \frac{1}{(1+\tan(u)^2)^{3/2}} \frac{1}{\cos(u)^2}\,du$$
$$= \int_0^{\pi/4} \frac{1}{(\frac{1}{\cos(u)^2})^{3/2}} \frac{1}{\cos(u)^2}\,du$$
$$= \int_0^{\pi/4} \cos(u)\,du$$
$$= \big[\sin(u)\big]_0^{\pi/4} = \sin(\pi/4) = \frac{1}{\sqrt{2}}.$$

7 **a)** Vis med delvis integrasjon at

$$\int \frac{\sin x}{x}\,dx = \frac{1-\cos x}{x} + \int \frac{1-\cos x}{x^2}\,dx.$$

b) Med utgangspunkt i formelen fra **a)**, vis at

$$\int_1^\infty \frac{\sin x}{x}\,dx$$

er konvergent.

Solution

a) We use integration by parts with $u = \frac{1}{x}$ and $v' = \sin(x)$, then $u' = \frac{-1}{x^2}$ and $v = 1-\cos(x)$. We get

$$\int \frac{\sin(x)}{x}\,dx = \int uv'\,dx = uv - \int u'v\,dx = \frac{1-\cos(x)}{x} + \int \frac{1-\cos(x)}{x^2}\,dx.$$

Note that the choice of $v = 1-\cos(x)$ was to match the desired form. Generally, we can always add a constant term to v when doing integration by parts. **b)** Based on the formula derived in **a)**, we can rewrite the integral as

$$\int_1^\infty \frac{\sin x}{x}\,dx = \lim_{t\to\infty}\left(\int_1^t \frac{1-\cos x}{x^2}\,dx + \frac{1-\cos x}{x}\bigg|_1^t\right).$$

Since $\frac{1-\cos(t)}{t}$ goes to zero as $t\to\infty$, we only need to show that $\int_1^\infty \frac{1-\cos x}{x^2}\,dx$ is convergent.

Notice that $0 \le 1-\cos(x) \le 2$, therefore, by the comparison test,

$$\int_1^\infty \frac{1-\cos(x)}{x^2}\,dx \le \int_1^\infty \frac{2}{x^2}\,dx < \infty.$$

Therefore, $\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent.