Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1101 Basic Calculus I
Fall 2022

Exercise set 8: Solutions

1 Differentiate the given functions below and simplify your answers if possible. Also state when the domain of the derivatives.

a) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto e^{(e^x)}$

b) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \dfrac{e^x}{1+e^x}$

c) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto 2^{(x^2-3x+8)}$

Hint: Use chain rule of differentiation.

Solution.

a) $f(x) = e^{(e^x)}$, $f'(x) = e^{(e^x)}e^x = e^{x+e^x}$, $x \in \mathbb{R}$.

b) $f(x) = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$, $f'(x) = \frac{e^x}{(1+e^x)^2}$, $x \in \mathbb{R}$.

c) $f(x) = 2^{(x^2-3x+8)}$, $f'(x) = (2x-3)(\ln 2)2^{(x^2-3x+8)}$, $x \in \mathbb{R}$.

2 Let a function given by $f(x) = Ae^x\cos(x) + Be^x\sin(x)$, where $x \in \mathbb{R}$, and A, B are real constants. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$.

Solution.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) =& Ae^x\cos(x) - Ae^x\sin(x) + Be^x\sin(x) + Be^x\cos(x)\\ =&(A+B)e^x\cos(x) + (B-A)e^x\sin(x).\end{aligned}$$

3 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big(Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)\Big)$ and use this to calculate the indefinite integrals

$$\int e^{ax}\cos(bx)\,\mathrm{d}x \quad \text{and} \quad \int e^{ax}\sin(bx)\,\mathrm{d}x.$$

Solution.

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\Big(Ae^{ax}\cos(bx)+Be^{ax}\sin(bx)\Big)\\ =&Aae^{ax}\cos(bx)-Abe^{ax}\sin(bx)+Bae^{ax}\sin(bx)+Bbe^{ax}\cos(bx)\\ =&(Aa+Bb)e^{ax}\cos(bx)+(Ba-Ab)e^{ax}\sin(bx).\end{aligned}$$

If $Aa+Bb=1$ and $Ba-Ab=0$, then $A=\frac{a}{a^2+b^2}$ and $B=\frac{b}{a^2+b^2}$. Thus

$$\int e^{ax}\cos(bx)\,\mathrm{d}x=\frac{1}{a^2+b^2}\Big(ae^{ax}\cos(bx)+be^{ax}\sin(bx)\Big)+C.$$

If $Aa+Bb=0$ and $Ba-Ab=1$, then $A=\frac{-b}{a^2+b^2}$ and $B=\frac{a}{a^2+b^2}$. Thus

$$\int e^{ax}\sin(bx)\,\mathrm{d}x=\frac{1}{a^2+b^2}\Big(ae^{ax}\sin(bx)-be^{ax}\cos(bx)\Big)+C.$$

4 Find the sum of the given series below, or show that the series diverge.

a) $\displaystyle\sum_{k=0}^{\infty}\frac{2^{k+3}}{e^{k-3}}$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots$

Hint: Use that $\displaystyle\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\Big(\frac{1}{2n-1}-\frac{1}{2n+1}\Big).$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}$

Solution.

a)

$$\sum_{k=0}^{\infty}\frac{2^{k+3}}{e^{k-3}}=8e^3\sum_{k=0}^{\infty}\Big(\frac{2}{e}\Big)^k=\frac{8e^3}{1-\frac{2}{e}}=\frac{8e^4}{e-2}.$$

b)

Let

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots.$$

Since

$$\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\Big(\frac{1}{2n-1}-\frac{1}{2n+1}\Big),$$

the partial sum is

$$\begin{aligned} s_n =& \frac{1}{2}\Big(1 - \frac{1}{3}\Big) + \frac{1}{2}\Big(\frac{1}{3} - \frac{1}{5}\Big) + \cdots + \frac{1}{2}\Big(\frac{1}{2n-3} - \frac{1}{2n-1}\Big) + \frac{1}{2}\Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big) \\ =& \frac{1}{2}\Big(1 - \frac{1}{2n+1}\Big). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim_{n\to\infty} s_n = \frac{1}{2}.$$

c)

Since $\frac{1}{2n-1} > \frac{1}{2n} = \frac{1}{2} \cdot \frac{1}{n}$, therefore the partial sums of the given series exceed half those of the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$. Hence the given series diverges to infinity.

5 Use problem 4b) to show that $\lim_{M,N\to\infty} \sum_{n=M}^{N} \frac{1}{n^2} = 0$, and thus that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Solution. The sum in problem 4b) can be written as

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}.$$

Thus for each n, it holds that $\frac{1}{4n^2} \leq \frac{1}{(2n-1)(2n+1)}$. We can hence write

$$\sum_{n=M}^{N} \frac{1}{n^2} \leq 4 \sum_{n=M}^{N} \frac{1}{4n^2-1} = 4(s_N - s_M)$$

where s_N is the n:th partial sum from problem 4b). This quantity goes to zero as $M, N \to \infty$ as can be seen by

$$s_N - s_M = \frac{1}{2}\left(1 - \frac{1}{2N+1} - 1 + \frac{1}{2M+1}\right) \to 0$$

as $M, N \to \infty$. The conclusion is that the partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ make up a Cauchy sequence and hence are convergent.

6 Decide whether the given statements are TRUE or FALSE. If it is TRUE, prove it. If it is FALSE, give a counterexample.

a) If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges to infinity.

b) If $a_n \geq c > 0$ for every n, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges to infinity.

c) If $a_n > 0$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$ converges.

Solution.

a)

FALSE. A counterexample is $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$. Clearly,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \lim_{n\to\infty} \frac{-\frac{1}{2}\left[1-\left(-\frac{1}{2}\right)^n\right]}{1-\left(-\frac{1}{2}\right)} = -\frac{1}{3}$$

is convergent. However, $\sum_{n=1}^{\infty} \frac{2^n}{(-1)^n}$ is oscillating to $-\infty$ and ∞ as $n \to \infty$. So it diverges, but not only diverges to infinity.

b)

TRUE. We have

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n \geq c + c + c + \cdots + c = nc,$$

and $nc \to \infty$ as $n \to \infty$.

c)

TRUE. Since $\sum_{n=1}^{\infty} a_n$ converges, therefore $\lim_{n\to\infty} a_n = 0$.

Thus there exists N such that $0 < a_n \leq 1$ for $n \geq N$. Thus $0 < a_n^2 \leq a_n$ for $n \geq N$.

If $S_n = \sum_{k=N}^{n} a_k^2$ and $s_n = \sum_{k=N}^{n} a_k$, then $\{S_n\}$ is increasing and bounded above:

$$S_n \leq s_n \leq \sum_{k=1}^{\infty} a_k < \infty.$$

Thus $\sum_{k=N}^{\infty} a_k^2$ converges, and so $\sum_{k=1}^{\infty} a_k^2$ converges.

7 The hyperbolic trigonometric functions are defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

In this exercise we will deduce properties of these functions.

a) Compute the first and second derivatives of $y(x) = \sinh(x)$. What can you say about the quantity

$$y''(x) - y(x)?$$

b) Show that

$$\cosh(x)^2 - \sinh(x)^2 = 1.$$

c) Find an expression for $\sinh^{-1}(x)$.

Solution.

a)

We compute

$$\frac{d}{dx}\sinh(x) = \frac{e^x}{2} - \frac{-e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x),$$
$$\frac{d}{dx}\cosh(x) = \frac{e^x}{2} + \frac{-e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh(x).$$

Thus

$$y''(x) - y(x) = \sinh(x) - \sinh(x) = 0.$$

b) We compute

$$\cosh(x)^2 - \sinh(x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = \frac{4}{4} = 1.$$

c) We solve $y = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$ for y as

$$2y = e^x - e^{-x} \implies 2ye^x = (e^x)^2 - 1.$$

Let $t = e^x$ so that the above can be written as

$$2yt = t^2 - 1 \implies t^2 - 2yt - 1 = 0 \implies t = y \pm \sqrt{y^2 + 1}.$$

Note that the solution corresponding to the negative root is negative which $t = e^x$ never is since we assume $x \in \mathbb{R}$. We therefore choose the positive root for the inverse and take the logarithm to get x alone as

$$x = \sinh^{-1}(y) = \ln\left(y + \sqrt{y^2 + 1}\right).$$

8 The function $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ is sometimes used as an activation function for neural networks due to its range, monoticity, smoothness and limit properties - let's verify these!

- Show that the derivative of tanh exists everywhere and is a continuous function.
- Show that tanh is an increasing function.
- Show that $\lim_{x\to\pm\infty}\tanh(x) = \pm 1$.
- Determine the range of tanh.

Solution.

- We compute the derivative as

$$\frac{d}{dx}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{4}{(e^x + e^{-x})^2}$$

  which exists everywhere as $e^x + e^{-x} > 0$ for all x. To see that this function is continuous, note that it is the composition of continuous functions as e^x is continuous.

- The function being increasing follows as a simple consequence of $\frac{d}{dx}\tanh(x) = \frac{4}{(e^x+e^{-x})^2} > 0$.

- By inspection, tanh is an odd function and so it suffices to verify the behavior as $x \to \infty$. We write

$$\lim_{x\to\infty} \tanh(x) = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1}{1} = 1.$$

- By oddness, it suffices to determine the range of tanh on $[0, \infty)$. We first note that $\tanh(x) < 1$ for all $x > 0$ since

$$\tanh(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

  by the above calculation. By combining this with the limit result, we conclude that for any $y \in [0, 1)$, we can find an $x \in [0, \infty)$ such that $y < \tanh(x) < 1$. It now follows by the intermediate value theorem that the range of tanh on $[0, \infty)$ is $[0, 1)$. Indeed, let $y \in [0, 1)$ and choose z such that $y < \tanh(z) < 1$. Applying the intermediate value theorem, we get existence of an $x \in (0, z)$ such that $\tanh(x) = y$.

  By reflecting to get the range on all of $\mathbb{R}$, we conclude that the range of tanh is $(-1, 1)$.