Norwegian University of Science and Technology
Department of Mathematical Sciences

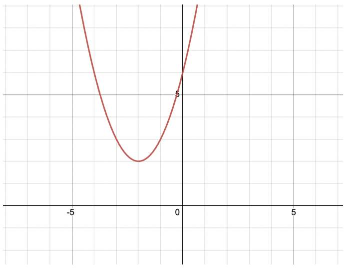
MA1101 Basic Calculus I
Fall 2022

Exercise set 1: Solutions

1 Sketch the graph of the following function

$$f(x) = (x+2)^2 + 2.$$

Solution.



2 Show that the *triangle inequality*

$$|a - b| \leq |a| + |b|$$

holds for all real numbers a and b.

Solution. Since both sides of the inequality are non-negative, it suffices to show that $|a+b|^2 \leq (|a|+|b|)^2$. Indeed,

$$|a+b|^2 = (a+b)^2 = a^2 + b^2 + 2ab = |a|^2 + |b|^2 + 2ab \leq |a|^2 + |b|^2 + 2|a||b| = (|a|+|b|)^2$$

where we used that $x \leq |x|$ for the $ab \leq |ab| = |a||b|$ step.

3 Find the roots of the following polynomial

$$x^4 + 8x^3 + 16x^2.$$

If a root is repeated, give its multiplicity. Also, write the polynomial as a product of linear factors.

Solution. $x^4 + 8x^3 + 16x^2 = x^2(x^2 + 8x + 16) = x^2(x+4)^2$. There are two double roots, 0 and -4.

4 Find the natural domain and range of the following functions.

a) $f(x) = x^3$

b) $f(x) = \sqrt{8-2x}$

c) $f(x) = \frac{1}{1-\sqrt{x-2}}$

Solution.

a) $f(x) = x^3$; natural domain $\mathbb{R}$, range $\mathbb{R}$.

b) $f(x) = \sqrt{8-2x}$; natural domain $(-\infty, 4]$, range $[0, \infty)$.

c) $f(x) = \frac{1}{1-\sqrt{x-2}}$; the denominator satisfies $x - 2 \geq 0$ and $1 - \sqrt{x-2} \neq 0$. Thus, natural domain $[2, 3) \cup (3, \infty)$. Firstly, the equation $y = f(x)$ can be solved for $x = 2 + \left(1 - \frac{1}{y}\right)^2$ so has a real solution provided $y \neq 0$. Secondly, when $x \in [2, 3)$, $0 < 1 - \sqrt{x-2} \leq 1$, so we have $f(x) = \frac{1}{1-\sqrt{x-2}} \geq 1$; when $x \in (3, \infty)$, $1 - \sqrt{x-2} < 0$, so we have $f(x) = \frac{1}{1-\sqrt{x-2}} < 0$. Thus, range $(-\infty, 0) \cup [1, \infty)$.

5 Suppose that $-x$ belongs to the domain of a function f whenever x does. We say that f is an **even function** if

$$f(-x) = f(x) \quad \text{for every } x \text{ in the domain of } f.$$

We say that f is an **odd function** if

$$f(-x) = -f(x) \quad \text{for every } x \text{ in the domain of } f.$$

What function f, defined on the real line $\mathbb{R}$, is both even and odd?

Solution. If f is both even and odd then $f(x) = f(-x) = -f(x)$, so $f(x) = 0$ identically.

6 Let $f(x) = \frac{2}{x}$ and $g(x) = \frac{x}{1-x}$. Construct the following composite functions and specify the natural domain of each.

a) $f \circ g$

b) $g \circ f$

Solution. a)

$$f \circ g = \frac{2}{\frac{x}{1-x}} = \frac{2(1-x)}{x};$$

natural domain is $\{x \in \mathbb{R} : x \neq 0, 1\}$.

b)

$$g \circ f = \frac{\frac{2}{x}}{1 - \frac{2}{x}} = \frac{2}{x-2};$$

natural domain is $\{x \in \mathbb{R} : x \neq 0, 2\}$.

7 Express $(\cos(3x))$ in terms of $(\sin(x))$ and $(\cos(x))$.
Hint: Begin by $3x = 2x + x$, and use sum formula of the cosine function.

Solution.

$$\begin{aligned}\cos(3x) =& \cos(2x + x)\\ =& \cos(2x)\cos(x) - \sin(2x)\sin(x)\\ =& (2\cos^2(x) - 1)\cos(x) - 2\sin^2(x)\cos(x)\\ =& 2\cos^3(x) - \cos(x) - 2(1 - \cos^2(x))\cos(x)\\ =& 4\cos^3(x) - 3\cos(x).\end{aligned}$$

8 A tangent function is defined by

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

where $x \in \mathbb{R}$ and $x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.
Specify the natural domain and prove the following identity

$$\frac{1 - \cos(x)}{1 + \cos(x)} = \tan^2\left(\frac{x}{2}\right).$$

Solution. To assure the above equality exists, we need

$$1 + \cos(x) \neq 0 \quad \text{and} \quad \frac{x}{2} \neq \frac{\pi}{2} + k\pi,$$

which gives the natural domain

$$\{x \in \mathbb{R} : x \neq \pi + 2k\pi\}.$$

Then

$$\frac{1 - \cos(x)}{1 + \cos(x)} = \frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)} = \tan^2\left(\frac{x}{2}\right).$$