MA1101
Grunnkurs i analyse I
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Department of Mathematical Sciences

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1 Hvilke av følgende utsagn er korrekte? Svar med «Sann» eller «Usann». *Begrunnelse trengs ikke på denne oppgaven.*

a) Det eksisterer to forskjellige løsninger til differensialligningen $y' = y$.

b) Det eksisterer to forskjellige løsninger til initialverdi-problemet $y' = y+1$, $y(0) = 2$.

c) For enhver kontinuerlig funksjon $f : \mathbb{R} \to \mathbb{R}$ gjelder det at

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(\frac{2n}{N} + 1\right) = \int_0^2 f(t)\,dt.$$

d) Enhver kontinuerlig funksjon $f\colon [0,5] \to \mathbb{R}$ er begrenset.

e) Enver begrenset funksjon $f\colon [0,2] \to \mathbb{R}$ har et maksimum.

f) Hvis $f\colon [0,1] \to \mathbb{R}$ er kontinuerlig og $f'(x) > 0$ for $x \in (0,1)$, så er $f(0) < f(1)$.

g) $\dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

h) Hvis A er en delmengde av $\mathbb{R}$ som har et minste verdi, så er $\inf A = \min A$.

i) Hvis A og B er delmengder av $\mathbb{R}$ og $C = \{x - y \,:\, x \in A,\ y \in B\}$, så er

$$\inf C = \inf A - \inf B.$$

j) $\lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = \infty.$

Solution.

a) True $(0, e^x)$

b) False (1st order equation, one initial condition)

c) False $(= \frac{1}{2}\int_1^3 f(t)dt)$

d) True (theorem from the course)

e) False (e.g. $f(x) = \begin{cases} x & x < 2 \\ 0 & x = 2. \end{cases}$)

f) True (mean value theorem)

g) True (provable by differentiating $x = \sin(\arcsin(x))$)

h) True (by definition)

i) False (e.g. $A = [0, 1], B = [0, 1]$)

j) False ($\to e$).

2 Løs initialverdiproblemet

$$y' = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}, \qquad y(0) = \frac{1}{\sqrt{2}},$$

og finn det største intervallet I (som inneholder 0) hvor løsningen eksisterer.

Solution.

The equation is separable which means that we can rewrite it as

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \implies \int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$$
$$\implies \arcsin(y) = \arcsin(x) + C.$$

Plugging in the initial condition $(x, y) = (0, \frac{1}{\sqrt{2}})$ yields $\arcsin(\frac{1}{\sqrt{2}}) = \arcsin(0) + C$ which yields $C = \frac{\pi}{4}$.

Taking the sine of both sides of $\arcsin(y) = \arcsin(x) + \frac{\pi}{4}$ yields

$$y = \sin(\arcsin(x) + \frac{\pi}{4}).$$

The original differential equation is only well-defined for $-1 < x < 1$, and the solution is valid for all such x. Therefore the sought interval I is $(-1, 1)$.

3 La

$$f(x) = \frac{x+2}{x^2+1} + 2\arctan(x).$$

a) Finn definisjonsmengden til f.

b) Bestem intervallene der f øker og avtar.

c) Finn asymptotene til f. (Husk at $\lim_{x\to\pm\infty} \arctan x = \pm\frac{\pi}{2}$.)

d) Avgjør om f antar et maksimum eller minimum.

Solution.

a) Since the denominator of the quotient is nonzero for all x and $\arctan(x)$ is defined for all x, we deduce that the domain of the function is all of $\mathbb{R}$.

b) We compute the derivative of f using $\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}$ as

$$f'(x) = \frac{x^2+1-2x(x+2)}{(x^2+1)^2} + \frac{2}{x^2+1} = \frac{x^2-4x+4}{(x^2+1)^2}.$$

Since the denominator is positive for all x, we determine the sign of the derivative by looking at the sign of x^2-4x+3 which is positive for $x \in (-\infty, 1) \cup (3, \infty)$ and negative for $x \in (1,3)$. Therefore f is increasing for $x \in (-\infty, 1] \cup [3, \infty)$ and decreasing on $[1,3]$.

c) Since the function has no singularities and the quotient goes to 0 as $|x|$ goes to infinite, the asymptotes are determined by $2\arctan(x)$ which are $y_1 = \pi$ and $y_2 = -\pi$.

d) From b), we know that the critical points of f are $x = 1, 3$. We need to determine if $f(1)$ or $f(3)$ are global extrema. We compute

$$f(1) = \frac{3}{2} + 2\arctan(1) = \frac{3}{2} + \frac{\pi}{2} < \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$
$$f(3) = \frac{5}{10} + 2\arctan(3) = \frac{1}{2} + \frac{\pi}{2} < \pi.$$

Since both of these values are less than π which we know that the function approaches as $x \to \infty$, we deduce that f does not attain a maximum. Similarly, f does not attain a minimum either since $f(x) > -\pi$ for all $x \in \mathbb{R}$ but $f(x) \to -\pi$ as $x \to -\infty$.

4 **a)** Beregn Taylorpolynomet av grad 4 rundt punktet $x = 0$ til funksjonen

$$f(x) = \int_0^x e^{-t^2+t}\, dt.$$

b) Finn en tilnærming til $f(0,1)$ med feil mindre enn $0,001$.

Solution.

a) We will use that we know the Taylor series expansion of e^x instead of computing values of derivatives of f at 0 (although that method also works). Since we will be integrating the Taylor series of e^{-t^2+t}, we only need to compute the degree 3 polynomial of e^{-t^2+t}. We do this as

$$\begin{aligned} e^{-t^2+t} &= 1 + (t-t^2) + \frac{1}{2}(t-t^2)^2 + \frac{1}{6}(t-t^2)^3 + \mathcal{O}(t^4) \\ &= 1 + t - t^2 + \frac{1}{2}(t^2 - 2t^3) + \frac{1}{6}t^3 = \mathcal{O}(t^4) \\ &= 1 + t - \frac{1}{2}t^2 - \frac{5}{6}t^3. \end{aligned}$$

Integrating, we find

$$f(x) = x + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{5}{24}x^4 + \mathcal{O}(x^5).$$

b) We use the degree 2 Taylor polynomial to approximate f and use that

$$|f(0,1) - (0,1 - \frac{0,1^2}{2})| = \frac{f'''(c)}{3!}0,1^3 \qquad \text{where } |c| \leq 0,1.$$

To bound the size of $f'''(c)$ for $|c| \leq 0,1$, we will need to compute the third derivative of f as

$$f'''(x) = e^{-x^2+x}(4x^2 - 4x - 1) \implies |f'''(c)| < 6$$

where we used that $|4c^2 - 4c - 1| \leq 2$ and $e^{-c^2-c} \leq e < 3$ for $|c| < 0,1$.

It now follows that the error $|f(0,1) - 0,1 - \frac{0,1^2}{2}| = |f(0,1) - 0.105|$ is bounded by $\frac{6}{6}(0,1)^3 = 0.001$ as desired.

5 **a)** Beregn $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{3 - x^2}\, dx$.

b) Beregn $\displaystyle\int \frac{\sin(\ln x)}{x}\, dx$.

c) Vis at $\displaystyle\int_1^{\infty} \frac{x^2 - 5}{x^4 + 3x + 2}\, dx$ konvergerer. *Hint: Bruk sammenligningstesten.*

Solution.

a) We make the substitution $x = \sqrt{3}\sin(u) \implies dx = \sqrt{3}\cos(u)\, du$ which yields

$$\begin{aligned}\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{3 - x^2}\, dx &= \int_{u=-\pi/2}^{u=\pi/2} \sqrt{3 - 3\sin^2(u)}\sqrt{3}\cos(u)\, du \\ &= 3\int_{-\pi/2}^{\pi/2} \cos^2(u)\, du.\end{aligned}$$

To compute this integral, we use the cosine double angle formula

$$\cos(2u) = 2\cos^2(u) - 1 \implies \cos^2(u) = \frac{\cos(2u) + 1}{2}$$

as

$$\begin{aligned}3\int_{-\pi/2}^{\pi/2} \cos^2(u)\, du &= \frac{3}{2}\int_{-\pi/2}^{\pi/2} \big(\cos(2u) + 1\big)\, du \\ &= \frac{3}{2}\Big[\frac{1}{2}\sin(2u) + u\Big]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}.\end{aligned}$$

b) We make the substitution $u = \ln(x) \implies du = \frac{1}{x}dx$ which yields

$$\begin{aligned}\int \frac{\sin(\ln(x))}{x}\,dx &= \int \frac{\sin(u)}{x} x\,du \\ &= \int \sin(u)\,du = -\cos(u) + C = -\cos(\ln(x)) + C.\end{aligned}$$

c) We bound the integrand as

$$\left|\frac{x^2-5}{x^4+3x+2}\right| \leq \frac{x^2+5x^2}{x^4+3x+2} \leq \frac{6x^2}{2x^4} = \frac{3}{x^2}.$$

Since

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^\infty = 1 < \infty,$$

the comparison test therefore shows that the integral is convergent.

6 **a)** Finn et eksempel på en funksjon $f\colon \mathbb{R} \to \mathbb{R}$ slik at $\lim_{n\to\infty} f(1/n) = 0$, men grenseverdien $\lim_{x\to 0} f(x)$ ikke eksisterer.

b) Finn et eksempel på en kontinuerlig funksjon $f\colon \mathbb{R} \to \mathbb{R}$ slik at $f'(0)$ ikke eksisterer.

c) Finn et eksempel på en to ganger kontinuerlig deriverbar funksjon $f\colon \mathbb{R} \to \mathbb{R}$ slik at den tredje deriverte $f'''(0)$ ikke eksisterer.

Solution.

a) Since f is not required to be continuous we can choose the function very freely. Two valid examples are

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q}, \end{cases} \qquad f(x) = \sin\left(\frac{\pi}{x}\right).$$

b) The standard example of this is $f(x) = |x|$ where the left and right limits defining the derivative at $x = 0$ differ.

c) We know that if we can find a function f such that $f''(x) = |x|$, we are done. To this end, we compute

$$\int |x|\,dx = \frac{x^2}{2}\operatorname{sign}(x), \qquad \int \frac{x^2}{2}\operatorname{sign}(x)\,dx = \frac{x^3}{6}\operatorname{sign}(x) = \frac{|x|^3}{6}.$$

And hence we can choose our function to be $f(x) = |x|^3$.

7 Anta at $f\colon \mathbb{R} \to \mathbb{R}$ er to ganger kontinuerlig deriverbar og at $|f''(x)| \leq 4$ for alle $x \in \mathbb{R}$.

a) Vis at

$$\lim_{x\to\infty} \frac{f(x)}{x^3} = 0.$$

b) Vis at det eksisterer positive konstanter A, B, C slik at

$$|f(x)| \leq A + B|x| + C|x|^2$$

for alle $x \in \mathbb{R}$.
Hint: Analysens fundamentalteorem.

Solution.

a) We apply L'hopital's rule twice to deduce that

$$\lim_{x\to\infty} \frac{f(x)}{x^3} = \lim_{x\to\infty} \frac{f''(x)}{6x}.$$

Since the numerator is uniformly bounded and the denominator goes to infinity as $x \to \infty$, we conclude that the limit must be 0.

b) This statement is stronger and does not follow from the L'hopital argument used above. Note first that since f is twice continuously differentiable,

$$|f'(x) - f'(0)| = \left| \int_0^x f''(t)\, dt \right| \leq \left| \int_0^x 4\, dx \right| = 4|x| \implies |f'(x)| \leq 4|x| + |f'(0)|$$

Similarly,

$$|f(x) - f(0)| = \left| \int_0^x f'(t)\, dt \right| \leq \left| \int_0^x \big(4t + |f'(0)|\big)\, dt \right| = 2x^2 + |f'(0)||x|$$
$$\implies |f(x)| \leq |f(0)| + |f'(0)||x| + 2|x|^2.$$

Hence, we can choose our constants as $A = |f(0)|$, $B = |f'(0)|$, $C = 2$.