Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1101
Grunnkurs i analyse I
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1 Hvilke av følgende utsagn er korrekte? Svar med «Sann» eller «Usann». *Begrunnelse trengs ikke på denne oppgaven.*

a) Funksjonen $x \mapsto |x|$ er deriverbar på hele $\mathbb{R}$.

b) Hvis $f\colon \mathbb{R} \to \mathbb{R}$ er kontinuerlig og $F(x) = \int_0^x f(t)\,dt$, så er F deriverbar på hele $\mathbb{R}$.

c) $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\frac{n}{N}) = \int_0^1 f(t)\,dt$ for enhver kontinuerlig funksjon $f : [0,1] \to \mathbb{R}$.

d) Enhver kontinuerlig funksjon $f\colon \mathbb{R} \to \mathbb{R}$ er begrenset.

e) Enver begrenset funksjon $f\colon \mathbb{R} \to \mathbb{R}$ er kontinuerlig.

f) Hvis $f\colon [0,1] \to \mathbb{R}$ er kontinuerlig og $f'(1/2) = 0$, så er $x = 1/2$ enten et minimumspunkt eller et maksimumspunkt til f.

g) $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$ for alle $x \in \mathbb{R}$.

h) Hvis A og B er delmengder av $\mathbb{R}$ og $C = \{x + y \,:\, x \in A,\ y \in B\}$, så er

$$\sup C = \sup A + \sup B.$$

i) Hvis $A = \{x \in \mathbb{R} \,:\, x^2 < 3\}$, så er $\sup A = 3$.

j) $\displaystyle\lim_{n\to\infty}\left(2 + \frac{1}{n}\right)^n = \infty$.

Solution.

a) False (not at $x = 0$).

b) True (Fundamental theorem of calculus)

c) True (Riemann sum)

d) False (e.g. $x \mapsto x$)

e) False (e.g. $x \mapsto \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$)

f) False (e.g. $x \mapsto (x - 1/2)^3$)

g) True (standard integral, verifiable)

h) True (Provable using standard argument)

i) False ($\sup A = \sqrt{3}$)

j) True ($\lim_{n\to\infty}(2+\frac{1}{n})^n \geq \lim_{n\to\infty} 2^n = \infty$)

2 Løs initialverdiproblemet

$$y' = x + xy^2, \qquad y(0) = 1,$$

og finn det største intervallet I (som inneholder 0) hvor løsningen eksisterer.

Solution. The differential equation is separable and can hence be solved as

$$\begin{aligned} \frac{dy}{dx} = x(1+y^2) &\implies \int \frac{1}{1+y^2}\,dy = \int x\,dx \\ &\implies \arctan(y) = \frac{1}{2}x^2 + C. \end{aligned}$$

Plugging in the initial condition $(x, y) = (0, 1)$ yields $\arctan(1) = 0 + C = \frac{\pi}{4}$.

Taking the tangent of both sides of $\arctan(y) = \frac{1}{2}x^2 + \frac{\pi}{4}$ yields

$$y = \tan(x^2/2 + \pi/4).$$

To determine the largest interval containing 0 in which the above is a solution, note first that the function $x \mapsto \tan(x)$ is discontinuous at $x = \pm\frac{\pi}{2}$ and infinitely differentiable between those points. Hence the interval in which the solution exists is the set of x for which

$$\left|\frac{x^2}{2} + \frac{\pi}{4}\right| < \frac{\pi}{2} \iff |x| < \sqrt{\frac{\pi}{2}}.$$

We conclude that the interval is $I = \left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$.

3 La

$$f(x) = e^{-x}(x^2 + 2x - 1), \quad -3 \leq x \leq 3.$$

Bestem alle lokale maksimums- og minimumspunkter til f på intervallet $[-3, 3]$. Bestem også funksjonens globale maksimum og minimum på $[-3, 3]$.

Solution. Recall that maximums and minimums are found at critical points, endpoints or singular points. We compute the derivative of f as

$$f'(x) = e^{-x}(3 - x^2)$$

which always exists and is zero only for $x = \pm\sqrt{3}$. Since there are no singular points we compute the values at the endpoints and critical points which yields

$$\begin{aligned} f(-3) &= 2e^3, \\ f(-\sqrt{3}) &= 2(1-\sqrt{3})e^{\sqrt{3}}, \\ f(\sqrt{3}) &= 2(1+\sqrt{3})e^{-\sqrt{3}}, \\ f(3) &= 14e^{-3} \end{aligned}$$

We now investigate the minimum/maximum properties of f at each of these points.

For $x = \pm\sqrt{3}$ we can compute the signs of the second derivative of f as

$$f''(x) = e^{-x}(x^2 - 2x - 3) \implies \begin{aligned} f''(-\sqrt{3}) &= 2\sqrt{3}e^{\sqrt{3}} > 0, \\ f''(\sqrt{3}) &= -2\sqrt{3}e^{-\sqrt{3}} < 0. \end{aligned}$$

This means that $x = -\sqrt{3}$ is a local minimum and $x = \sqrt{3}$ is a local maximum.

For the endpoints $x = \pm 3$, we compute the derivatives as

$$f'(-3) = -6e^3 < 0 \qquad f'(3) = -6e^{-3} < 0.$$

Since at the left endpoint $x = -3$, the function is decreasing we deduce that $x = -3$ is a local maximum. Meanwhile at the right endpoint $x = 3$, the function is also decreasing and so $x = 3$ is a local minimum.

To find the *global* maximum and minimum, we compare all the values and deduce that the global maximum is attained at $x = -3$ and the global minimum is attained at $x = -\sqrt{3}$.

4 **a)** Beregn Taylorpolynomet av grad 3 rundt punktet $x = 0$ til funksjonen

$$f(x) = \int_0^x \sqrt{1+t^2}\, dt.$$

b) Finn en tilnærming til $f(0,1)$ med feil mindre enn $0,001$.

Solution.

a) We compute the derivatives and values at $x = 0$ of f as

$$\begin{aligned} f(x) &= \int_0^x \sqrt{1+t^2}\, dt \implies f(0) = 0, \\ f'(x) &= \sqrt{1+x^2} \implies f'(0) = 1, \\ f''(x) &= \frac{x}{\sqrt{1+x^2}} \implies f''(0) = 0, \\ f'''(x) &= \frac{1}{(1+x^2)^{3/2}} \implies f'''(0) = 1. \end{aligned}$$

Hence the Taylor series is of the form

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x + \frac{x^3}{6}.$$

b) Recall that by Taylor's theorem,

$$f(x) - P_2(x) = \frac{f'''(c)}{3!}x^3$$

for some c between 0 and x. For $x = 0.1$, we can bound this as

$$|f(x) - P_2(x)| = \frac{\frac{1}{(1+c^2)^{3/2}}}{6}0.1^3 \leq \frac{1}{6}0.1^3 = \frac{1}{6}0.001 < 0.001.$$

Hence $P_2(0.1)$ approximates $f(0.1)$ with error less than 0.001 as requested. The approximation is

$$P_2(0.1) = 0 + 0.1 + 0 = 0.1 \implies |f(0.1) - 0.1| < 0.001.$$

5 **a)** Beregn $\displaystyle\int_0^{1/2} \frac{dx}{(1-x^2)^{3/2}}$

b) Beregn $\displaystyle\int xe^{-x}\, dx.$

c) Vis at $\displaystyle\int_1^{\infty} e^{-x^2}\, dx$ konvergerer. *Hint: Bruk sammenligningstesten.*

Solution.

a) We make the substitution

$$x = \sin(u) \implies dx = \cos(u)du$$

which yields

$$\begin{aligned}\int_0^{1/2} \frac{dx}{(1-x^2)^{3/2}} &= \int_{x=0}^{x=1/2} \frac{\cos(u)}{(1-\sin^2(u))^{3/2}}\, du \\ &= \int_{x=0}^{x=1/2} \frac{1}{\cos^2(u)}\, du.\end{aligned}$$

At this point we recognize $\frac{1}{\cos^2(u)}$ as the derivative of $\tan(u)$ which gives us

$$\begin{aligned}\int_0^{1/2} \frac{dx}{(1-x^2)^{3/2}} &= \Big[\tan(u)\Big]_{x=0}^{x=1/2} = \left[\frac{\sin(u)}{\cos(u)}\right]_{x=0}^{x=1/2} \\ &= \left[\frac{x}{\sqrt{1-x^2}}\right]_0^{1/2} = \frac{1/2}{1-(\frac{1}{2})^2} = \frac{2}{3}.\end{aligned}$$

b) We use integration by parts with the two functions $x \mapsto x$ and $x \mapsto e^{-x}$ to obtain

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -e^{-x}(1+x).$$

c) We compare with the function $x \mapsto e^{-x}$ since $e^{-x^2} \leq e^{-x}$ for all $x \in [1, \infty)$. This yields

$$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty e^{-x}\,dx = \left[-e^{-x}\right]_1^\infty = 0 + e^{-1} < \infty.$$

6 **a)** Finn to positive følger $(a_n)_{n=1}^\infty$ og $(b_n)_{n=1}^\infty$ slik at

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \qquad \text{og} \qquad \lim_{n\to\infty}(a_n - b_n) = \infty.$$

b) La $(a_n)_{n=1}^\infty$ og $(b_n)_{n=1}^\infty$ være to positive følger slik at $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ og $\lim_{n\to\infty}(a_n - b_n) = \infty$. Bestem en følge $(c_n)_{n=1}^\infty$ (gitt ved følgene a_n og b_n) slik at både

$$\lim_{n\to\infty}(a_n - c_n) = \infty \quad \text{og} \quad \lim_{n\to\infty}(c_n - b_n) = \infty$$

holder samtidig.

c) La $(a_n)_{n=1}^\infty$ og $(b_n)_{n=1}^\infty$ være to positive følger slik at $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ og $\lim_{n\to\infty} a_n/b_n = \infty$. Bestem en følge $(c_n)_{n=1}^\infty$ (gitt ved følgene a_n og b_n) slik at både

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \infty \quad \text{og} \quad \lim_{n\to\infty} \frac{c_n}{b_n} = \infty$$

holder samtidig.

Solution.

c) One solution is to put $a_n = n^2 + n$ and $b_n = n^2$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} = 1$$

while

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty}(n^2 + n - n^2) = \lim_{n\to\infty} n = \infty.$$

b) Let $c_n = \frac{a_n+b_n}{2}$, the arithmetic mean of a_n and b_n. Then

$$\lim_{n\to\infty} c_n - a_n = \lim_{n\to\infty} \frac{a_n + b_n - 2a_n}{2} = \lim_{n\to\infty} \frac{b_n - a_n}{2} = 0$$

and

$$\lim_{n\to\infty} b_n - c_n = \lim_{n\to\infty} \frac{2b_n - a_n - b_n}{2} = \lim_{n\to\infty} \frac{b_n - a_n}{2} = 0.$$

c) Let $c_n = \sqrt{a_n b_n}$, the geometric mean of a_n and b_n. Then

$$\lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} \frac{\sqrt{a_n}}{\sqrt{b_n}} = \sqrt{\lim_{n\to\infty} \frac{a_n}{b_n}} = 0$$

and

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{a_n}}{\sqrt{b_n}} = \sqrt{\lim_{n\to\infty} \frac{a_n}{b_n}} = 0.$$

7 Anta at $f\colon \mathbb{R} \to \mathbb{R}$ er to ganger kontinuerlig deriverbar i $x = x_0$, og la

$$g(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0, \\ f'(x_0) & \text{if } x = x_0. \end{cases}$$

Vis at $g'(x_0) = \frac{f''(x_0)}{2}$.

Solution.

We compute $g'(x_0)$ as

$$\begin{aligned} g'(x_0) &= \lim_{h\to 0} \frac{g(x_0+h) - g(x_0)}{h} = \lim_{h\to 0} \frac{\frac{f(x_0+h)-f(x_0)}{x_0+h-x_0} - f'(x_0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)}{h} \\ &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - hf'(x_0)}{h^2}. \end{aligned}$$

Since the quantity is indeterminate of the form $\left[\frac{0}{0}\right]$ at $h = 0$, we use L'Hopital's rule to compute

$$\begin{aligned} \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - hf'(x_0)}{h^2} &= \lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0)}{2h} = \frac{1}{2} f''(x_0) \\ &= \frac{1}{2} \lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0)}{h} = \frac{1}{2} f''(x_0) \end{aligned}$$

which is the desired conclusion.