Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1101 Basic Calculus I
Fall 2022

Exercise set 8
Deadline: Oct. 21

You may write solutions in Norwegian or English, as preferable. The most important part is *how* you arrive at an answer, not the answer itself.

1 Differentiate the given functions below and simplify your answers if possible. Also state when the domain of the derivatives.

a) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto e^{(e^x)}$

b) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \dfrac{e^x}{1+e^x}$

c) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto 2^{(x^2-3x+8)}$

Hint: Use chain rule of differentiation.

2 Let a function given by $f(x) = Ae^x\cos(x) + Be^x\sin(x)$, where $x \in \mathbb{R}$, and A, B are real constants. Find $\dfrac{\mathrm{d}}{\mathrm{d}x} f(x)$.

3 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big(Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)\Big)$ and use this to calculate the indefinite integrals

$$\int e^{ax}\cos(bx)\,\mathrm{d}x \quad \text{and} \quad \int e^{ax}\sin(bx)\,\mathrm{d}x.$$

4 Find the sum of the given series below, or show that the series diverge.

a) $\displaystyle\sum_{k=0}^{\infty} \frac{2^{k+3}}{e^{k-3}}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots$

Hint: Use that $\dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2}\Big(\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\Big)$.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

5 Use problem 4b) to show that $\lim_{M,N\to\infty} \sum_{n=M}^{N} \frac{1}{n^2} = 0$, and thus that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

6 Decide whether the given statements are TRUE or FALSE. If it is TRUE, prove it. If it is FALSE, give a counterexample.

a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges to infinity.

b) If $a_n \geq c > 0$ for every n, then $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

c) If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (a_n)^2$ converges.

7 The hyperbolic trigonometric functions are defined as

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

In this exercise we will deduce properties of these functions.

a) Compute the first and second derivatives of $y(x) = \sinh(x)$. What can you say about the quantity

$$y''(x) - y(x)?$$

b) Show that

$$\cosh(x)^2 - \sinh(x)^2 = 1.$$

c) Find an expression for $\sinh^{-1}(x)$.

8 The function $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ is sometimes used as an activation function for neural networks due to its range, monoticity, smoothness and limit properties - let's verify these!

- Show that the derivative of tanh exists everywhere and is a continuous function.
- Show that tanh is an increasing function.
- Show that $\lim_{x\to\pm\infty} \tanh(x) = \pm 1$.
- Determine the range of tanh.