MA1101 Basic Calculus I
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Exercise set 7: Solutions

1 Show that the functions f below are bijective, and calculate the inverse functions f^{-1}. Specify the domains and ranges of f^{-1}.

a) $f\colon [1,\infty) \to \mathbb{R}, \quad x \mapsto \sqrt{x-1}$

b) $f\colon (-\infty,-1)\cup(-1,\infty) \to \mathbb{R}, \quad x \mapsto \dfrac{x}{1+x}$

Solution.

In this two questions, we denote $\mathscr{D}(f)$ and $\mathscr{R}(f)$ as the domain and range for a function f.

a)

$f(x) = \sqrt{x-1}$.

$$\begin{aligned} f(x_1) = f(x_2) &\Longleftrightarrow \sqrt{x_1 - 1} = \sqrt{x_2 - 1}, \quad (x_1, x_2 \geq 1) \\ &\Longleftrightarrow x_1 - 1 = x_2 - 1 = 0 \\ &\Longleftrightarrow x_1 = x_2. \end{aligned}$$

Thus f is bijective. Let $y = f^{-1}(x)$.

Then $x = f(y) = \sqrt{y-1}$, and $y = 1 + x^2$. Thus $f^{-1}(x) = 1 + x^2$, $(x \geq 0)$.

$\mathscr{D}(f^{-1}) = \mathscr{R}(f) = [0,\infty)$, $\mathscr{R}(f^{-1}) = \mathscr{D}(f) = [1,\infty)$.

b)

$f(x) = \dfrac{x}{1+x}$.

If $f(x_1) = f(x_2)$, then $\frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}$. Hence $x_1(1+x_2) = x_2(1+x_1)$, and on simplification, $x_1 = x_2$. Thus, f is bijective.

Let $y = f^{-1}(x)$. Then $x = f(y) = \dfrac{y}{1+y}$ and $x(1+y) = y$. Thus $y = \dfrac{x}{1-x} = f^{-1}(x)$.

$\mathscr{D}(f^{-1}) = \mathscr{R}(f) = (-\infty,1)\cup(1,\infty)$, $\mathscr{R}(f^{-1}) = \mathscr{D}(f) = (-\infty,-1)\cup(-1,\infty)$.

2 Differentiate the given functions below and simplify your answers if possible. Also state when the domain of the derivatives.

a) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto e^{(e^x)}$

b) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \dfrac{e^x}{1+e^x}$

c) $f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto 2^{(x^2-3x+8)}$

Hint: Use chain rule of differentiation.

Solution.

a) $f(x) = e^{(e^x)}$, $f'(x) = e^{(e^x)}e^x = e^{x+e^x}$, $x \in \mathbb{R}$.

b) $f(x) = \frac{e^x}{1+e^x} = 1 - \frac{1}{1+e^x}$, $f'(x) = \frac{e^x}{(1+e^x)^2}$, $x \in \mathbb{R}$.

c) $f(x) = 2^{(x^2-3x+8)}$, $f'(x) = (2x-3)(\ln 2)2^{(x^2-3x+8)}$, $x \in \mathbb{R}$.

3 Find the value of x when

$$2^{x^2-3} = 4^x.$$

Solution.

$2^{x^2-3} = 4^x = 2^{2x} \implies x^2 - 3 = 2x$.

$x^2 - 2x - 3 = 0 \implies (x-3)(x+1) = 0$. Hence, $x = -1$ or 3.

4 Let a function given by $f(x) = Ae^x\cos(x) + Be^x\sin(x)$, where $x \in \mathbb{R}$, and A, B are real constants. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$.

Solution.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) =& Ae^x\cos(x) - Ae^x\sin(x) + Be^x\sin(x) + Be^x\cos(x)\\ =& (A+B)e^x\cos(x) + (B-A)e^x\sin(x).\end{aligned}$$

5 Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\Big(Ae^{ax}\cos(bx) + Be^{ax}\sin(bx)\Big)$ and use this to calculate the indefinite integrals

$$\int e^{ax}\cos(bx)\,\mathrm{d}x \quad \text{and} \quad \int e^{ax}\sin(bx)\,\mathrm{d}x.$$

Solution.

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\Big(Ae^{ax}\cos(bx)+Be^{ax}\sin(bx)\Big)\\ =&Aae^{ax}\cos(bx)-Abe^{ax}\sin(bx)+Bae^{ax}\sin(bx)+Bbe^{ax}\cos(bx)\\ =&(Aa+Bb)e^{ax}\cos(bx)+(Ba-Ab)e^{ax}\sin(bx).\end{aligned}$$

If $Aa+Bb=1$ and $Ba-Ab=0$, then $A=\frac{a}{a^2+b^2}$ and $B=\frac{b}{a^2+b^2}$. Thus

$$\int e^{ax}\cos(bx)\,\mathrm{d}x=\frac{1}{a^2+b^2}\Big(ae^{ax}\cos(bx)+be^{ax}\sin(bx)\Big)+C.$$

If $Aa+Bb=0$ and $Ba-Ab=1$, then $A=\frac{-b}{a^2+b^2}$ and $B=\frac{a}{a^2+b^2}$. Thus

$$\int e^{ax}\sin(bx)\,\mathrm{d}x=\frac{1}{a^2+b^2}\Big(ae^{ax}\sin(bx)-be^{ax}\cos(bx)\Big)+C.$$

6 If functions f and g have respective inverse f^{-1} and g^{-1}, show that the composite function $f\circ g$ has inverse $(f\circ g)^{-1}=g^{-1}\circ f^{-1}$.

Solution.

If $y=(f\circ g)^{-1}(x)$, then $x=f\circ g(y)=f(g(y))$. Thus $g(y)=f^{-1}(x)$ and $y=g^{-1}(f^{-1}(x))=g^{-1}\circ f^{-1}(x)$. That is, $(f\circ g)^{-1}=g^{-1}\circ f^{-1}$.

7 Find the sum of the given series below, or show that the series diverge.

a) $\displaystyle\sum_{k=0}^{\infty}\frac{2^{k+3}}{e^{k-3}}$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots$

Hint: Use that $\displaystyle\frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\Big(\frac{1}{2n-1}-\frac{1}{2n+1}\Big).$

c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}$

Solution.

a)

$$\sum_{k=0}^{\infty}\frac{2^{k+3}}{e^{k-3}}=8e^3\sum_{k=0}^{\infty}\Big(\frac{2}{e}\Big)^k=\frac{8e^3}{1-\frac{2}{e}}=\frac{8e^4}{e-2}.$$

b)

Let

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots .$$

Since

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big),$$

the partial sum is

$$\begin{aligned} s_n =& \frac{1}{2}\Big(1-\frac{1}{3}\Big) + \frac{1}{2}\Big(\frac{1}{3}-\frac{1}{5}\Big) + \cdots + \frac{1}{2}\Big(\frac{1}{2n-3} - \frac{1}{2n-1}\Big) + \frac{1}{2}\Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big) \\ =& \frac{1}{2}\Big(1 - \frac{1}{2n+1}\Big). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim_{n\to\infty} s_n = \frac{1}{2}.$$

c)

Since $\frac{1}{2n-1} > \frac{1}{2n} = \frac{1}{2}\cdot\frac{1}{n}$, therefore the partial sums of the given series exceed half those of the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}$. Hence the given series diverges to infinity.

8 Find the required Taylor series representations of the functions below.

a) $f(x) = e^{-2x}$ about -1

b) $f(x) = \cos^2(x)$ about $\dfrac{\pi}{8}$

Solution.

a)

Let $t = x+1$, so $x = t-1$. We have

$$f(x) = e^{-2x} = e^{-2(t-1)} = e^2 \sum_{n=0}^{\infty} \frac{(-2)^n t^n}{n!} = e^2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (x+1)^n}{n!} \quad \text{(for all } x\text{)}.$$

b)

let $y = x - \frac{\pi}{8}$; then $x = y + \frac{\pi}{8}$. Thus,

$$\begin{aligned}
&\cos^2(x) = \cos^2\left(y + \frac{\pi}{8}\right) = \frac{1}{2}\Big[1 + \cos\Big(2y + \frac{\pi}{4}\Big)\Big] \\
=&\frac{1}{2}\Big[1 + \frac{1}{\sqrt{2}}\cos(2y) - \frac{1}{\sqrt{2}}\sin(2y)\Big] \\
=&\frac{1}{2} + \frac{1}{2\sqrt{2}}\Big[1 - \frac{(2y)^2}{2!} + \frac{(2y)^4}{4!} - \cdots\Big] - \frac{1}{2\sqrt{2}}\Big[2y - \frac{(2y)^3}{3!} + \frac{(2y)^5}{5!} - \cdots\Big] \\
=&\frac{1}{2} + \frac{1}{2\sqrt{2}}\Big[1 - 2y - \frac{(2y)^2}{2!} + \frac{(2y)^3}{3!} + \frac{(2y)^4}{4!} - \frac{(2y)^5}{5!} - \cdots\Big] \\
=&\frac{1}{2} + \frac{1}{2\sqrt{2}}\Big[1 - 2\Big(x - \frac{\pi}{8}\Big) - \frac{2^2}{2!}\Big(x - \frac{\pi}{8}\Big)^2 + \frac{2^3}{3!}\Big(x - \frac{\pi}{8}\Big)^3 + \frac{2^4}{4!}\Big(x - \frac{\pi}{8}\Big)^4 \\
&- \frac{2^5}{5!}\Big(x - \frac{\pi}{8}\Big)^5 - \cdots\Big] \\
=&\frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\left[\frac{2^{2n-1}}{(2n-1)!}\Big(x - \frac{\pi}{8}\Big)^{2n-1} + \frac{2^{2n}}{(2n)!}\Big(x - \frac{\pi}{8}\Big)^{2n}\right] \quad \text{(for all } x\text{)}.
\end{aligned}$$

9 Decide whether the given statements are TRUE or FALSE. If it is TRUE, prove it. If it is FALSE, give a counterexample.

a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges to infinity.

b) If $a_n \geq c > 0$ for every n, then $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

c) If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (a_n)^2$ converges.

Solution.

a)

FALSE. A counterexample is $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$. Clearly,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \lim_{n\to\infty} \frac{-\frac{1}{2}\Big[1 - \big(-\frac{1}{2}\big)^n\Big]}{1 - \big(-\frac{1}{2}\big)} = -\frac{1}{3}$$

is convergent. However, $\sum_{n=1}^{\infty} \frac{2^n}{(-1)^n}$ is oscillating to $-\infty$ and ∞ as $n \to \infty$. So it diverges, but not only diverges to infinity.

b)

TRUE. We have

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n \geq c + c + c + \cdots + c = nc,$$

and $nc \to \infty$ as $n \to \infty$.

c)

TRUE. Since $\sum_{n=1}^{\infty} a_n$ converges, therefore $\lim_{n\to\infty} a_n = 0$.

Thus there exists N such that $0 < a_n \leq 1$ for $n \geq N$. Thus $0 < a_n^2 \leq a_n$ for $n \geq N$.

If $S_n = \sum_{k=N}^{n} a_k^2$ and $s_n = \sum_{k=N}^{n} a_k$, then $\{S_n\}$ is increasing and bounded above:

$$S_n \leq s_n \leq \sum_{k=1}^{\infty} a_k < \infty.$$

Thus $\sum_{k=N}^{\infty} a_k^2$ converges, and so $\sum_{k=1}^{\infty} a_k^2$ converges.