MA1101 Basic Calculus I
Fall 2021

Exercise set 6: Solutions

Norwegian University of Science and Technology
Department of Mathematical Sciences

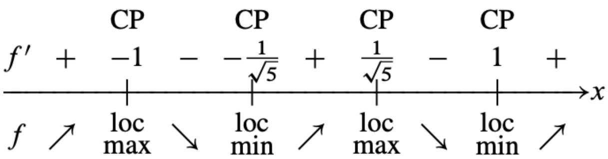
1 Classify the critical points of the function

$$f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x(x^2-1)^2.$$

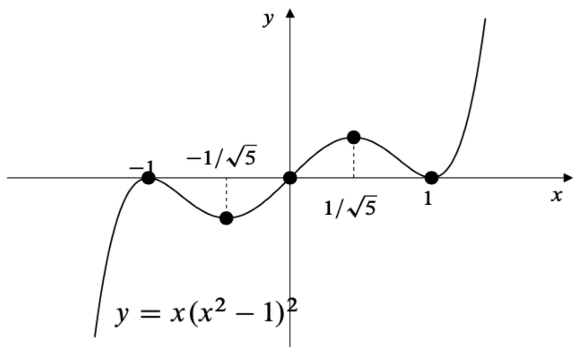
(That is, decide if they are local/global maxim/minima or not.)

Solution.

$$\begin{aligned} f(x) =& x(x^2-1)^2, \\ f'(x) =& (x^2-1)^2 + 2x(x^2-1)2x \\ =& (x^2-1)(x^2-1+4x^2) \\ =& (x^2-1)(5x^2-1) \\ =& (x-1)(x+1)(\sqrt{5}x-1)(\sqrt{5}x+1). \end{aligned}$$



$f(\pm 1) = 0$, $f(\pm\frac{1}{\sqrt{5}}) = \pm\frac{16}{25\sqrt{5}}$.



Thus,

$$-1, \quad \text{local maximum point}; \quad -\frac{1}{\sqrt{5}} \quad \text{local minimum point};$$
$$+\frac{1}{\sqrt{5}} \quad \text{local maximum point}; 1, \quad \text{local minimum point};$$
$$\text{There are no global maximum or minimum point.}$$

2 Sketch the graph of the function

$$f\colon \mathbb{R}\backslash\{\pm 1\} \to \mathbb{R}, \quad x \mapsto \frac{x^3}{x^2-1}.$$

Make a table with the sign of f' and f'', and the corresponding behavior of f. Describe the asymptotes of f.

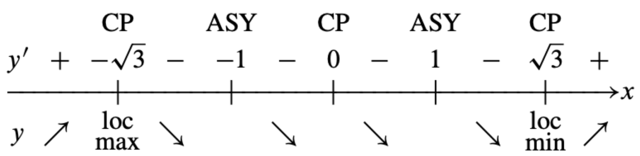
Solution.

$$f(x) = \frac{x^3}{x^2-1}, \quad f'(x) = \frac{x^2(x^2-3)}{(x^2-1)^2}, \quad f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}.$$

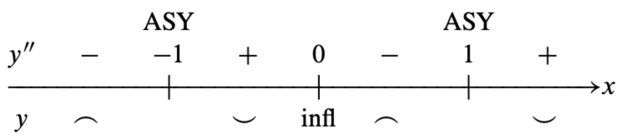
From f : Intercept: $(0,0)$. Asymptotes: $x = \pm 1$ (vertical), $y = x$ (oblique). Symmetry: odd. Other points: $\left(\pm\sqrt{3}, \pm\frac{3\sqrt{3}}{2}\right)$.

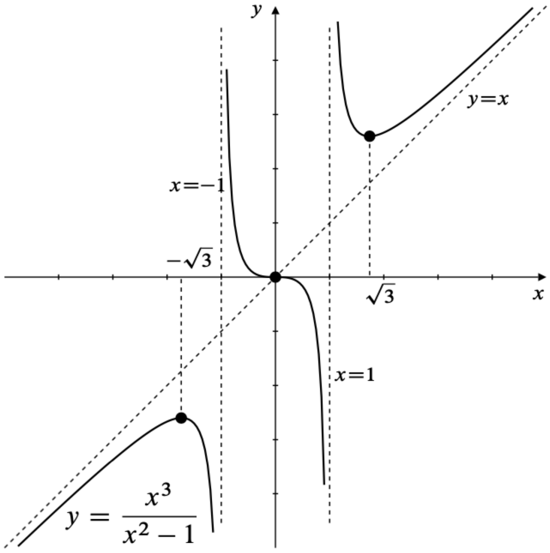
For computing the oblique asymptotes: We may write $f(x) = x + \frac{x}{x^2-1}$. When x tends to $\pm\infty$, it behaves like the $y = x$.

From $f'(x)$: Critical point: $x = 0, \pm\sqrt{3}$.



From $f''(x)$: $f''(x) = 0$ at $x = 0$.





3 All 80 rooms in a motel will be rented each night if the manager charges 40 NOK or less per room. If he charges $(40 + x)$ NOK per room, then $2x$ rooms will remain vacant. If each rented room costs the manager 10 NOK per day and each unrented room 2 NOK per day in overhead, how much should the manager charge per room to maximize his daily profit?

Solution.

If the manager charges $(40 + x)$ NOK per room, then $(80 - 2x)$ rooms will be rented.

The total income will be $(80 - 2x)(40 + x)$ NOK and the total cost will be $(80 - 2x)(10) + (2x)(2)$ NOK. Therefore, the profit is

$$\begin{aligned} P(x) =&(80 - 2x)(40 + x) - [(80 - 2x)(10) + (2x)(2)] \\ =&2400 + 16x - 2x^2, \quad \text{for} \quad x > 0. \end{aligned}$$

If $P'(x) = 16 - 4x = 0$, then $x = 4$. Since $P''(x) = -4 < 0$, P must have a maximum value at $x = 4$. Therefore, the manager should charge 44 NOK per room.

4 Find the linearization of the given function about the given point.

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sqrt{3 + x^2} \quad \text{about} \quad x = 1.$$

Solution.

$$f(1) = 2, \quad f'(x) = \frac{1}{2}(3 + x^2)^{-\frac{1}{2}} \cdot 2x = x(3 + x^2)^{-\frac{1}{2}}, \quad f'(1) = \frac{1}{2}.$$

Thus, the linearization of f about $x = 1$ is

$$L(x) = 2 + \frac{1}{2}(x - 1) = \frac{x}{2} + \frac{3}{2}.$$

5 Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial. Show that the Taylor series around $x_0 = 0$ of $p(x)$ is equal to $p(x)$.

Solution.

$$\begin{aligned} p^{(1)}(x) =& a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + \cdots + a_2 2x + a_1, \\ p^{(2)}(x) =& a_n n(n-1)x^{n-2} + a_{n-1}(n-1)(n-2)x^{n-3} + \cdots + a_3 \cdot 3 \cdot 2x + 2a_2, \\ &\vdots \\ p^{(n)}(x) =& a_n n(n-1)\cdots(n-(n-1)) = a_n n!, \\ p^{(n+1)}(x) =& 0, \\ &\vdots \end{aligned}$$

Then we get

$$\begin{aligned} &p^{(1)}(0) = 1!a_1, \quad p^{(2)}(0) = 2!a_2, \quad p^{(3)}(0) = 3!a_3, \quad \cdots, \quad p^{(n)}(0) = n!a_n, \\ &p^{(n+1)}(0) = 0, \quad p^{(n+2)}(0) = 0, \quad \cdots \end{aligned}$$

By Taylor's expansion, we get

$$\begin{aligned} p(x) =& p(0) + \frac{p'(0)}{1!}(x-0) + \frac{p''(0)}{2!}(x-0)^2 + \cdots + \frac{p^{(n)}(0)}{n!}(x-0)^n + \cdots \\ =& a_0 + \frac{1!a_1}{1!}x + \frac{2!a_2}{2!}x^2 + \frac{3!a_3}{3!}x^3 + \cdots \frac{n!a_n}{n!}x^n + 0 + 0 + \cdots \\ =& a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0. \end{aligned}$$

6 Find the fourth order Taylor polynomial of the function $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$ at the point $x_0 = \ln 2$.

Solution.

$$f^{(k)}(x) = e^x; \quad f^{(k)}(x_0) = e^{\ln 2}.$$

Thus,

$$\begin{aligned} P_4(x) =& e^{\ln 2} + e^{\ln 2}(x - \ln 2) + \frac{e^{\ln 2}}{2!}(x-\ln 2)^2 + \frac{e^{\ln 2}}{3!}(x-\ln 2)^3 + \frac{e^{\ln 2}}{4!}(x-\ln 2)^4 \\ =& 2 + 2(x-\ln 2) + (x-\ln 2)^2 + \frac{1}{3}(x-\ln 2)^3 + \frac{1}{12}(x-\ln 2)^4. \end{aligned}$$

7 Calculate

a) $\lim_{x\to\infty} \dfrac{3x + \ln(x) + 2x^3}{x^3}$.

b) $\lim_{x\to 0} \dfrac{\tan(x) - x}{x^2 \sin(x)}$.

Solution.

a) Using L'Hôpital's rule, we have

$$\lim_{x\to\infty} \frac{3x + \ln(x) + 2x^3}{x^3} = \lim_{x\to\infty} \frac{3 + \frac{1}{x} + 6x^2}{3x^2} = 2.$$

b)

Method 1: Using L'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{x\to0} \frac{\tan(x) - x}{x^2\sin(x)} = \lim_{x\to0} \frac{\sec^2(x) - 1}{2x\sin(x) + x^2\cos(x)} = \lim_{x\to0} \frac{\frac{1}{\cos^2(x)} - 1}{2x\sin(x) + x^2\cos(x)} \\
&= \lim_{x\to0} \frac{\sin^2(x)}{x\cos^2(x)\left[2\sin(x) + x\cos(x)\right]} = \lim_{x\to0} \frac{\sin(x)}{2\sin(x) + x\cos(x)} \\
&= \lim_{x\to0} \frac{\cos(x)}{2\cos(x) + \cos(x) - x\sin(x)} = \frac{1}{3}.
\end{aligned}$$

Method 2: Using Taylor's expansion, we have

$$\tan(x) = x + \frac{1}{3}x^3 + o(x^3).$$

Thus,

$$\lim_{x\to0} \frac{\tan(x) - x}{x^2\sin(x)} = \lim_{x\to0} \frac{\frac{1}{3}x^3 + o(x^3)}{x^2\sin(x)} = \frac{1}{3}.$$

8 Calculate

a) $\lim_{x\to0} \frac{\sin(ax)}{\sin(bx)}$, for $a, b > 0$.

b) $\lim_{h\to0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$, given that f is twice derivable.

Solution.

a) Method 1: Using L'Hôpital's rule, we have

$$\lim_{x\to0} \frac{\sin(ax)}{\sin(bx)} = \lim_{x\to0} \frac{a\cos(ax)}{b\cos(bx)} = \frac{a}{b}.$$

Method 2:

$$\lim_{x\to0} \frac{\sin(ax)}{\sin(bx)} = \lim_{x\to0} \frac{\frac{\sin(ax)}{ax} \cdot ax}{\frac{\sin(bx)}{bx} \cdot bx} = \frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b}.$$

b)

Method 1: Using L'Hôpital's rule twice, we get

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$
$$= \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2} = \frac{2f''(x)}{2} = f''(x).$$

Method 2: By Taylor's expansion, we have

$$\begin{aligned}
f(x+h) =& f(x)+f'(x)(x+h-x)+\frac{f''(x)}{2}(x+h-x)^2+o\big((x+h-x)^2\big)\\
=& f(x)+f'(x)h+\frac{f''(x)}{2}h^2+o(h^2);\\
f(x-h) =& f(x)+f'(x)(x-h-x)+\frac{f''(x)}{2}(x-h-x)^2+o\big((x-h-x)^2\big)\\
=& f(x)-f'(x)h+\frac{f''(x)}{2}h^2+o(h^2).
\end{aligned}$$

Then we have

$$f(x+h)+f(x-h) = 2f(x)+f''(x)h^2+o(h^2).$$

Thus, we get

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = \lim_{h\to 0}\frac{2f(x)+f''(x)h^2+o(h^2)-2f(x)}{h^2} = f''(x).$$

9 Prove Darboux's theorem: Let I be a closed interval, $f\colon I\to\mathbb{R}$ a real-valued differentiable function. Then f' has the intermediate value property: If a and b are points in I with $a<b$, then for every y between $f'(a)$ and $f'(b)$, there exists an x in $[a,b]$ such that $f'(x)=y$.

Hint: Prove it according to the following steps:

1. Consider the particular case, that is, y equals to $f'(a)$ or $f'(b)$;

2. Consider the general case. Without loss of generality, assume $f'(b)<y<f'(a)$. Define the auxiliary function $\varphi(t)=f(t)-yt$;

3. Apply the extreme value theorem to φ;

4. Analyse the sign of $\varphi'(a)$ and $\varphi'(b)$, and whether φ can attain its maximum value at a or b to get the desired conclusion.

Proof. If y equals $f'(a)$ or $f'(b)$, then setting x equal to a or b, respectively, gives the desired result.

Now assume that y is strictly between $f'(a)$ and $f'(b)$, and in particular that $f'(b)<y<f'(a)$. Let $\varphi\colon I\to\mathbb{R}$ such that $\varphi(t)=f(t)-yt$. If it is the case that $f'(a)<y<f'(b)$, we adjust our below proof, instead asserting that φ has its minimum on $[a,b]$.

Since φ is continuous on the closed interval $[a, b]$, the maximum value of φ on $[a, b]$ is attained at some point in $[a, b]$, according to the extreme value theorem.

Because $\varphi'(a) = f'(a) - y > 0$, we know φ cannot attain its maximum value at a. (If it did, then $\dfrac{\varphi(t) - \varphi(a)}{t - a} \leq 0$ for all $t \in (a, b]$, which implies $\varphi'(a) \leq 0$.)

Likewise, because $\varphi'(b) = f'(b) - y < 0$, we know φ cannot attain its maximum value at b.

Therefore, φ must attain its maximum value at some point $x \in (a, b)$. Hence, we have $\varphi'(x) = 0$, i.e. $f'(x) = y$. $\square$