Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1101 Basic Calculus I
Fall 2021

Exercise set 5: Solutions

1 Let $y\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{a+bx}$, where $a \neq 0$, $b \neq 0$.

a) Find $\frac{\mathrm{d}^3 y}{(\mathrm{d}x)^3}(x)$.

b) Find a general formula for $\frac{\mathrm{d}^n y}{(\mathrm{d}x)^n}(x)$ for $n \in \mathbb{N}$. Give an argument for it.

Solution.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= (-1)b(a+bx)^{-2} = (-1)^1 \cdot 1! \cdot b^1 (a+bx)^{-1-1};\\
\frac{\mathrm{d}^2 y}{(\mathrm{d}x)^2} &= (-1)(-2)b^2(a+bx)^{-3} = (-1)^2 \cdot 2! \cdot b^2 (a+bx)^{-2-1};\\
\frac{\mathrm{d}^3 y}{(\mathrm{d}x)^3} &= (-1)(-2)(-3)b^3(a+bx)^{-4} = (-1)^3 \cdot 3! \cdot b^3 (a+bx)^{-3-1}\\
&\vdots\\
\frac{\mathrm{d}^n y}{(\mathrm{d}x)^n} &= (-1)^n \cdot n! \cdot b^n (a+bx)^{-n-1}. \qquad (1)
\end{aligned}
$$

So the answer of **a)** is $\frac{\mathrm{d}^3 y}{(\mathrm{d}x)^3} = (-1)(-2)(-3)b^3(a+bx)^{-4}$.

For **b)**, let us use mathematical induction to show (1) is correct. When $n = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -b(a+bx)^{-2} = (-1)^1 \cdot 1! \cdot b^1(a+bx)^{-1-1}$. The case $k = 1$ is true. Assume (1) is true for $k = n-1$, that is, $\frac{\mathrm{d}^{n-1}y}{(\mathrm{d}x)^{n-1}} = (-1)^{n-1} \cdot (n-1)! \cdot b^{n-1}(a+bx)^{-n}$. Then we take the derivative one more time to $\frac{\mathrm{d}^{n-1}y}{(\mathrm{d}x)^{n-1}}$ to get

$$
\frac{\mathrm{d}^n y}{(\mathrm{d}x)^n} = (-1)^{n-1} \cdot (n-1)! \cdot b^{n-1} \cdot (-n)b(a+bx)^{-n-1} = (-1)^n \cdot n! \cdot b^n (a+bx)^{-n-1},
$$

which gives (1).

2 Show that

$$
\sin(2x) > x \quad \text{when} \quad 0 < x < \frac{\pi}{8}.
$$

Prove it analytically, not graphically.

Solution.

Since $0 < x < \frac{\pi}{8}$, by the Mean-Value Theorem, there exists $c \in (0, 2x)$, such that

$$\frac{\sin(2x) - \sin(0)}{2x - 0} = \frac{\mathrm{d}}{\mathrm{d}x} \sin(x)\Big|_{x=c} = \cos(c).$$

Since $0 < c < 2x < \frac{\pi}{4}$, we have $0 < c < \frac{\pi}{4}$, so $\frac{\sqrt{2}}{2} < \cos(c) < 1$. Thus,

$$\frac{\sin(2x) - \sin(0)}{2x - 0} > \frac{\sqrt{2}}{2}, \quad \sin(2x) > \sqrt{2}x > x, \quad 0 < x < \frac{\pi}{8}.$$

3 The volume V in a water tank can be described using the formula

$$V(t) = 350(20 - t)^2\,\mathrm{L}, \quad t \geq 0.$$

The relevant physical unit is liters (L), and we count $t = 0$ as the start time. The time t is measured in minutes. How much water flows out per minute after 5 minutes; after 15 minutes?

Solution.

Volume in tank is $V(t) = 350(20 - t)^2\,\mathrm{L}$ at t min. At $t = 5$, water volume is changing at rate

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{t=5} = -700(20 - t)\bigg|_{t=5} = -10500.$$

Water is draining out at $10500\,\mathrm{L/min}$ at that time.

At $t = 15$, water volume is changing at rate

$$\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{t=15} = -700(20 - t)\bigg|_{t=15} = -3500.$$

Water is draining out at $3500\,\mathrm{L/min}$ at that time.

4 Show that the function $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is increasing on the whole real line even though $f'(x)$ is not positive at every point.

Solution.

$f(x) = x^3$ is increasing on $(-\infty, 0)$ and $(0, \infty)$ because $f'(x) = 3x^2 > 0$ there. But $f(x_1) < f(0) = 0 < f(x_2)$ whenever $x_1 < 0 < x_2$, so f is also increasing on intervals containing the origin.

5 Use (formal) implicit differentiation to find the tangent to the curve (x, y) when

$$x^2 + y^2 + 2xy + x = 1, \quad (x, y) = (0, 1).$$

Solution.

$$\begin{aligned} x^2 + y^2 + 2xy + x &= 1, \\ 2x + 2yy' + 2y + 2xy' + 1 &= 0. \end{aligned}$$

At $(x, y) = (0, 1)$, $2y' + 2 + 1 = 0$, so the slope is $y' = -\frac{3}{2}$. Thus, the equation of the tangent line is

$$y - 1 = -\frac{3}{2}x, \quad \text{or} \quad 3x + 2y - 2 = 0.$$

6 Let $z\colon \mathbb{R}\backslash A \to \mathbb{R}$, $x \mapsto \tan\left(\frac{x}{2}\right)$, where $A = \{x\colon x = \frac{\pi}{4} + \frac{k\pi}{2},\, k \in \mathbb{Z}\}$. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{2}{1+z^2}, \quad \sin(x) = \frac{2z}{1+z^2}, \quad \text{and} \quad \cos(x) = \frac{1-z^2}{1+z^2}.$$

Solution.

If $z = \tan\left(\frac{x}{2}\right)$, then

$$1 = \sec^2\left(\frac{x}{2}\right)\frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{1 + \tan^2\left(\frac{x}{2}\right)}{2}\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{1+z^2}{2}\frac{\mathrm{d}x}{\mathrm{d}z}.$$

Thus, $\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{2}{1+z^2}$. Also

$$\begin{aligned} \cos(x) &= 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2\left(\frac{x}{2}\right)} - 1 = \frac{2}{1+z^2} - 1 = \frac{1-z^2}{1+z^2}; \\ \sin(x) &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{2z}{1+z^2}. \end{aligned}$$

7 Determine

$$\int \frac{2x}{\sqrt{x^2+1}}\,\mathrm{d}x.$$

Solution.

Since $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x^2+1} = \frac{x}{\sqrt{x^2+1}}$, therefore

$$\int \frac{2x}{\sqrt{x^2+1}}\,\mathrm{d}x = 2\sqrt{x^2+1} + C.$$

8 Let $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $x \mapsto x - \frac{4}{x}$. Show that $f(-1) = f(4)$, but that there is no point $c \in (-1, 4)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

Solution.

A direct computation shows that

$$f(-1) = -1 + 4 = 3, \quad f(4) = 4 - 1 = 3, \quad \text{so} \quad f(-1) = f(4).$$

Firstly, we see that when $x \to 0^+$, $f(x) \to -\infty$; when $x \to 0^-$, $f(x) \to +\infty$, which means at $x = 0$, the function is not continuous. Secondly, $f'(x) = 1 + \frac{4}{x^2} > 0$. This shows that on $(-\infty, 0) \cup (0, +\infty)$, f is strictly increasing and it does not have a critical point also on $(-1, 4)$. Thus, there is no point $c \in (-1, 4)$ such that $f'(c) = 0$.

Recall that the condition of Rolle's Theorem requires that the function is continuous on $[a, b]$ and differentiable on (a, b). By the above analysis, f is not continuous, and not differentiable at $x = 0$. Rolle's Theorem can not be applied in such case. So this does not contradict Rolle's Theorem.

9 Locate any inflection points of the given function below.

$$f(x) = \frac{x^3}{3} - 4x^2 + 12x - \frac{25}{3}, \quad x \in \mathbb{R}.$$

Solution.

$$f'(x) = x^2 - 8x + 12, \quad f''(x) = 2x - 8.$$

When $x = 4$, $f''(x) = 0$. On the interval $(-\infty, 4)$, $f''(x) < 0$; on the interval $(4, +\infty)$, $f''(x) > 0$. So the point $(4, -3)$ is the inflection point of $f(x)$.