Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1101 Basic Calculus I
Fall 2021

Exercise set 4: Solutions

1 Find the equations of the form $y = kx + m$ tangent at x_0 to

a) $y = \frac{1}{\sqrt{x}}, \quad x_0 = 9$

b) $y = \frac{1}{x^2+1}, \quad x_0 = 0.$

Solution.

a) The slope of $y = \frac{1}{\sqrt{x}}$ at $x_0 = 9$ is

$$\begin{aligned} k &= \lim_{h\to 0} \frac{1}{h}\Big(\frac{1}{\sqrt{9+h}} - \frac{1}{3}\Big) \\ &= \lim_{h\to 0} \frac{3-\sqrt{9+h}}{3h\sqrt{9+h}} \cdot \frac{3+\sqrt{9+h}}{3+\sqrt{9+h}} \\ &= \lim_{h\to 0} \frac{9-9-h}{3h\sqrt{9+h}(3+\sqrt{9+h})} \\ &= -\frac{1}{54}. \end{aligned}$$

The tangent line at $(9, \frac{1}{3})$ is $y = \frac{1}{3} - \frac{1}{54}(x-9)$, i.e. $y = -\frac{1}{54}x + \frac{1}{2}$.

b) The slope of $y = \frac{1}{x^2+1}$ at $x_0 = 0$ is

$$k = \lim_{h\to 0} \frac{1}{h}\Big(\frac{1}{h^2+1} - 1\Big) = \lim_{h\to 0} \frac{-h}{h^2+1} = 0.$$

The tangent line at $(0, 1)$ is $y = 1$.

2 Calculate, using the definition, the derivative of the following functions

a) $f\colon \mathbb{R} \to \mathbb{R}, \qquad x \mapsto 1 + 4x - 5x^2$

b) $g\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}, \qquad x \mapsto x + \frac{1}{x}$

c) $h\colon (-1, \infty) \to (0, \infty), \qquad x \mapsto \frac{1}{\sqrt{1+x}}.$

Also determine the maximal domain for the functions f', g' and h'.

Solution.

a)

$$\begin{aligned} f(x) =& 1 + 4x - 5x^2. \\ f'(x) =& \lim_{h\to 0} \frac{1 + 4(x+h) - 5(x+h)^2 - (1 + 4x - 5x^2)}{h} \\ =& \lim_{h\to 0} \frac{4h - 10xh - 5h^2}{h} = 4 - 10x. \end{aligned}$$

The maximal domain is $x \in \mathbb{R}$.

b)

$$\begin{aligned} g(x) =& x + \frac{1}{x}. \\ g'(x) =& \lim_{h\to 0} \frac{x + h + \frac{1}{x+h} - x - \frac{1}{x}}{h} \\ =& \lim_{h\to 0} \left(1 + \frac{x - x - h}{h(x+h)x}\right) \\ =& 1 + \lim_{h\to 0} \frac{-1}{(x+h)x} = 1 - \frac{1}{x^2}. \end{aligned}$$

The maximal domain is $x \in \mathbb{R}\backslash\{0\}$.

c)

$$\begin{aligned} h(x) =& \frac{1}{\sqrt{1+x}}. \\ h'(x) =& \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+x+h}} - \frac{1}{\sqrt{1+x}}}{h} \\ =& \lim_{h\to 0} \frac{\sqrt{1+x} - \sqrt{1+x+h}}{h\sqrt{1+x+h}\sqrt{1+x}} \\ =& \lim_{h\to 0} \frac{1 + x - 1 - x - h}{h\sqrt{1+x+h}\sqrt{1+x}(\sqrt{1+x} + \sqrt{1+x+h})} \\ =& \lim_{h\to 0} -\frac{1}{\sqrt{1+x+h}\sqrt{1+x}(\sqrt{1+x} + \sqrt{1+x+h})} \\ =& -\frac{1}{2(1+x)^{\frac{3}{2}}}. \end{aligned}$$

The maximal domain is $x > -1$.

3 Show that the curve $y = x^2$ and the straight line $x + 4y = 18$ intersect at right angles at one of their two intersection points.

Hint: Find the product of their slopes at their intersection points.

Solution.

The intersection points of $y = x^2$ and $x + 4y = 18$ satisfy

$$\begin{aligned} 4x^2 + x - 18 =& 0 \\ (4x+9)(x-2) =& 0. \end{aligned}$$

Therefore $x = -\frac{9}{4}$ or $x = 2$.

The slope of $y = x^2$ is $k_1 = 2x$.

At $x = -\frac{9}{4}$, $k_1 = -\frac{9}{2}$. At $x = 2$, $k_1 = 4$.

The slope of $x + 4y = 18$, i.e. $y = -\frac{1}{4}x + \frac{18}{4}$, is $k_2 = -\frac{1}{4}$.

Thus, at $x = 2$, the product of these slopes is $4 \cdot (-\frac{1}{4}) = -1$. So, the curve and line intersect at right angles at that point.

4 Use the factoring of a difference of cubes

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

to show that

$$\frac{\mathrm{d}}{\mathrm{d}x} x^{\frac{1}{3}} = \frac{1}{3} x^{-\frac{2}{3}}, \quad x \neq 0,$$

with the help of the definition of derivative.

Solution.

$$\begin{aligned} f(x) =& x^{\frac{1}{3}}. \\ f'(x) =& \lim_{h \to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} \\ =& \lim_{h \to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} \\ & \times \frac{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} x^{\frac{1}{3}} + x^{\frac{2}{3}}}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} x^{\frac{1}{3}} + x^{\frac{2}{3}}} \\ =& \lim_{h \to 0} \frac{x + h - x}{h[(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} x^{\frac{1}{3}} + x^{\frac{2}{3}}]} \\ =& \lim_{h \to 0} \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} x^{\frac{1}{3}} + x^{\frac{2}{3}}} \\ =& \frac{1}{3} x^{-\frac{2}{3}}, \quad x \neq 0. \end{aligned}$$

5 Find

a) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\pi}{2-\pi t}\right), \quad t \neq \frac{2}{\pi}$

b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2+4)(\sqrt{x}+1)(5x^{\frac{2}{3}}-2)\right], \quad x > 0$

c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{2x+\frac{1}{3x+1}}\right)\Big|_{x=1}$.

Solution.

a)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\pi}{2-\pi t}\Big) = -\frac{\pi}{(2-\pi t)^2}(-\pi) = \frac{\pi^2}{(2-\pi t)^2}, \quad t \neq \frac{2}{\pi}.$$

b)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2+4)(\sqrt{x}+1)(5x^{\frac{2}{3}}-2)\right] \\
=&2x(\sqrt{x}+1)(5x^{\frac{2}{3}}-2) + \frac{1}{2\sqrt{x}}(x^2+4)(5x^{\frac{2}{3}}-2) + \frac{10}{3}x^{-\frac{1}{3}}(x^2+4)(\sqrt{x}+1) \\
=&\frac{95}{6}x^{\frac{13}{6}} - 4x + \frac{70}{3}x^{\frac{1}{6}} + \frac{40}{3}x^{\frac{5}{3}} - 5x^{\frac{3}{2}} + \frac{40}{3}x^{-\frac{1}{3}} - 4x^{-\frac{1}{2}}.
\end{aligned}$$

c)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{x}{2x+\frac{1}{3x+1}}\Big)\Big|_{x=1} = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{3x^2+x}{6x^2+2x+1}\Big)\Big|_{x=1} \\
=&\frac{(6x^2+2x+1)(6x+1)-(3x^2+x)(12x+2)}{(6x^2+2x+1)^2}\Big|_{x=1} = \frac{6x+1}{(6x^2+2x+1)^2}\Big|_{x=1} = \frac{7}{81}.
\end{aligned}$$

6 Find values of a and b that make

$$f(x) = \begin{cases} ax+b, & x < 0 \\ 2\sin x + 3\cos x, & x \geq 0 \end{cases}$$

differentiable at $x = 0$.

Solution.

f will be differentiable at $x = 0$ if

$$\begin{aligned} 2\sin 0 + 3\cos 0 = b, \quad \text{and} \\ \frac{\mathrm{d}}{\mathrm{d}x}(2\sin x + 3\cos x)\Big|_{x=0} = a. \end{aligned}$$

Thus, we need $b = 3$ and $a = 2$.

7 Find

a) $\frac{\mathrm{d}}{\mathrm{d}x}(2+|x|^3)^{\frac{1}{3}}$

b) $\frac{\mathrm{d}}{\mathrm{d}t}f\Big(2-3f(4-5t)\Big), \quad f\colon \mathbb{R}\to\mathbb{R}$ arbitrary

c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\Big|_{x=-2}$

and state in **a)**, **b)** and **c)** for which values of x applies.

Solution.

a)

$$\frac{\mathrm{d}}{\mathrm{d}x}(2+|x|^3)^{\frac{1}{3}} = \frac{1}{3}(2+|x|^3)^{-\frac{2}{3}}(3|x|^2)\mathrm{sgn}(x) = |x|^2(2+|x|^3)^{-\frac{2}{3}}\Big(\frac{x}{|x|}\Big)$$
$$= x|x|(2+|x|^3)^{-\frac{2}{3}}, \quad x\in\mathbb{R}.$$

b)

$$\frac{\mathrm{d}}{\mathrm{d}t}f\Big(2-3f(4-5t)\Big) = f'(2-3f(4-5t))\big(-3f'(4-5t)\big)(-5)$$
$$= 15f'(4-5t)f'(2-3f(4-5t)), \quad t\in\mathbb{R}.$$

c)

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{\sqrt{x^2-1}}{x^2+1}\Big)\Big|_{x=-2} = \frac{(x^2+1)\frac{x}{\sqrt{x^2-1}} - \sqrt{x^2-1}(2x)}{(x^2+1)^2}\Big|_{x=-2}$$
$$= \frac{2}{25\sqrt{3}}.$$

8 Calculate

a) $\lim_{x\to\pi}\sec(1+\cos(x))$. A secant function is defined by $\sec(x)=\frac{1}{\cos(x)}$, where $x\in\mathbb{R}$ and $x\neq\frac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$.

b) $\lim_{x\to 0}\cos\left(\frac{\pi-\pi\cos^2(x)}{x^2}\right)$.

Solution.

a)

$$\lim_{x\to\pi}\sec(1+\cos(x)) = \sec(1-1) = \sec(0) = 1.$$

b)

$$\lim_{x\to 0}\cos\left(\pi\Big(\frac{\sin(x)}{x}\Big)^2\right) = -1.$$

9 Assume $f(x)$ is continuous at $x = 0$. For the following statements, if TRUE, give reasons; if FALSE, give a counterexample.

a) If $\lim_{x\to 0} \frac{f(x)}{x}$ exists, then $f(0) = 0$.

b) If $\lim_{x\to 0} \frac{f(x)+f(-x)}{x}$ exists, then $f(0) = 0$.

c) If $\lim_{x\to 0} \frac{f(x)}{x}$ exists, then $f'(0)$ exists.

d) If $\lim_{x\to 0} \frac{f(x)-f(-x)}{x}$ exists, then $f'(0)$ exists.

Solution.

In **a)** and **b)**, since the limitation of denominator is 0, so the limit of numerator must also be 0, then we have $f(0) = 0$. Thus **a)** and **b)** are TRUE.

In **c)**, $\lim_{x\to 0} \frac{f(x)}{x}$ exists, then $f(0) = 0$, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$, so $f'(0)$ exists. **c)** is TRUE.

In **d)**, take $f(x) = |x|$, then

$$\lim_{x\to 0} \frac{f(x)-f(-x)}{x} = 0,$$

but $f(x)$ is not differentiable at $x = 0$. So **d)** is FALSE.