Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1101 Basic Calculus I
Fall 2021

Exercise set 13: Solutions

1 Compute $\lim_{n\to\infty} x_n$ for the following sequences or explain why the limit does not exist.

a)

$$x_n = \begin{cases} \frac{n+2}{n+1}, & n \text{ is odd} \\ \frac{n}{n+1}, & n \text{ is even} \end{cases}$$

b)

$$x_n = \begin{cases} \frac{n}{1+n}, & n \text{ is odd} \\ \frac{n}{1-n}, & n \text{ is even} \end{cases}$$

c)

$$x_n = \begin{cases} 1+\frac{1}{n}, & n \text{ is odd} \\ (-1)^n, & n \text{ is even} \end{cases}$$

d)

$$x_n = \begin{cases} 1, & n < 10^6 \\ \frac{1}{n}, & n \geq 10^6 \end{cases}$$

Solution.

a) When n is odd, $\lim_{n\to\infty} x_n = 1$; when n is even, $\lim_{n\to\infty} x_n = 1$. So $\lim_{n\to\infty} x_n = 1$.

b) When n is odd, $\lim_{n\to\infty} x_n = 1$; when n is even, $\lim_{n\to\infty} x_n = -1$. By the uniqueness of limit, $\lim_{n\to\infty} x_n$ does not exist.

c) $\lim_{n\to\infty} x_n = 1$.

d) $\lim_{n\to\infty} x_n = 0$. The limit of the sequence is not related to the former finite terms.

2 Evaluate the integral below.

a)

$$\int \frac{1}{5-x^2}\, dx$$

b)

$$\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\,dx$$

Hint: Let $x = a\tan(\theta)$.

c)

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx$$

Hint: Multiply by $\frac{\cos x}{\cos x}$ *and use* $u = \sin x$.

Solution.

a)

$$\begin{aligned}\frac{1}{5-x^2} =& \frac{A}{\sqrt{5}-x} + \frac{B}{\sqrt{5}+x}\\ =& \frac{(A+B)\sqrt{5}+(A-B)x}{5-x^2}\\ \Longrightarrow& \begin{cases} A+B = \frac{1}{\sqrt{5}}, \\ A-B=0 \end{cases} \quad \Longrightarrow A = B = \frac{1}{2\sqrt{5}}.\end{aligned}$$

Then

$$\begin{aligned}\int \frac{1}{5-x^2}\,dx =& \frac{1}{2\sqrt{5}}\int \Big(\frac{1}{\sqrt{5}-x} + \frac{1}{\sqrt{5}+x}\Big)\,dx\\ =& \frac{1}{2\sqrt{5}}\Big(-\ln|\sqrt{5}-x| + \ln|\sqrt{5}+x|\Big) + C\\ =& \frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5}+x}{\sqrt{5}-x}\right| + C.\end{aligned}$$

b)

Let $x = a\tan(\theta)$, then $dx = a\sec^2(\theta)\,d\theta$. Thus,

$$\begin{aligned}\int \frac{1}{(a^2+x^2)^{\frac{3}{2}}}\,dx =& \int \frac{a\sec^2(\theta)\,d\theta}{[a^2+a^2\tan^2(\theta)]^{\frac{3}{2}}}\\ =& \int \frac{a\sec^2(\theta)}{a^3\sec^3(\theta)}\,d\theta\\ =& \frac{1}{a^2}\int \cos(\theta)\,d\theta = \frac{1}{a^2}\sin(\theta) + C = \frac{x}{a^2\sqrt{a^2+x^2}} + C.\end{aligned}$$

c) Since $\cos\frac{\pi}{2} = 0$, this is an improper integral.

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\,dx = \lim_{C\to(\frac{\pi}{2})-} \int_0^C \frac{1}{\cos x}\,dx$$
$$= \lim_{C\to(\frac{\pi}{2})-} \int_0^C \frac{\cos x}{\cos^2 x}\,dx$$
$$\lim_{C\to(\frac{\pi}{2})-} \int_0^C \frac{\cos x}{1-\sin^2 x}\,dx$$

Substituting $u = \sin x$ and using partial fractions to integrate yields:

$$\lim_{C\to(\frac{\pi}{2})-} \ln\left|\frac{1+\sin x}{1-\sin x}\right|\Bigg|_0^C = \lim_{C\to(\frac{\pi}{2})-} \ln\left|\frac{1+\sin C}{1-\sin C}\right| = \infty.$$

This integral diverges to infinity.

3 Let

$$f(x) = \begin{cases} \frac{x}{3}\sin(\frac{2}{x}), & x < 0, \\ a, & x = 0, \\ \frac{2}{x}\sin(\frac{x}{3}), & x > 0. \end{cases}$$

Show that f is discontinuous at $x = 0$ no matter what the value of a is.

Hint: Consider $\lim_{x\to 0+} f(x)$ *and* $\lim_{x\to 0-} f(x)$.

Solution.

$\lim_{x\to 0-} f(x) = 0$, $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\sin(\frac{x}{3})}{\frac{x}{3}} \cdot \frac{2}{3} = \frac{2}{3}$. Since the limits from left and right are not equal, the function is discontinuous.

4 Find the sum of the given series, or show that the series diverges.

a)

$$\sum_{n=0}^{\infty} \frac{1}{e^n}$$

b)

$$\sum_{n=0}^{\infty} \frac{n}{n+2}$$

Solution.

a)

$$\sum_{n=0}^{\infty} \frac{1}{e^n} = 1 + \frac{1}{e} + \frac{1}{e^2} + \cdots = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}.$$

b) $\sum_{n=0}^{\infty} \frac{n}{n+2}$ diverges to infinity since $\frac{n}{n+2} = 1 > 0$.

5 Locate any inflection points of the given function below.

$$f(x) = \int_0^x (1-t)\arctan(t)\,dt.$$

Solution.

$$f'(x) = (1-x)\arctan(x).$$

let $f'(x) = 0$, then $x = 1$ and $x = 0$. Then

x	(-∞, 0)	0	(0, 1)	1	(1, ∞)
f'(x)	-	0	+	0	-

Thus, $x = 1$ is the maximal point of $f(x)$, and $x = 0$ is the minimal point.

6 Use the mean value theorem for integrals to calculate the limit below.

$$\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x}\,dx.$$

Solution.

By the mean value theorem for integrals, we have

$$\int_0^1 \frac{x^n}{1+x}\,dx = \frac{1}{1+\xi}\int_0^1 x^n\,dx, \quad 0 \le \xi \le 1.$$

Note that

$$\lim_{n\to\infty} \int_0^1 x^n\,dx = \lim_{n\to\infty} \frac{1}{n+1} = 0, \quad \text{and} \quad \frac{1}{2} \le \frac{1}{1+\xi} \le 1.$$

Thus,

$$\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x}\,dx = 0.$$

7 Let $f \in C([0,3],\mathbb{R})$ be differentiable on the open interval $(0,3)$, and $f(0)+2f(1)+3f(2)=6$, $f(3)=1$. Show that there exists $\xi \in (0,3)$, such that $f'(\xi)=0$.

Hint: Consider the minimum and maximum of f on interval $[0,2]$, then use the intermediate value theorem to show there exists $c \in [0,2]$, such that $f(c)=1$. To get some intuition draw a couple of examples, or start with the case where $f(x)$ is positive on the interval $[0,2]$.

Solution.

Since f is continuous on $[0,3]$, there exists minimum m and maximum M of f on $[0,2]$. By

$$6m \le f(0)+2f(1)+3f(2) \le 6M,$$

we have

$$m \le 1 \le M.$$

By the intermediate value theorem, there exists $c \in [0,2]$, such that $f(c)=1$. Since $f(c)=f(3)=1$, by Rolle's theorem, there exists $\xi \in (c,3) \subset (0,3)$, such that $f'(\xi)=0$.

8 *Old exam problem.*

a) Solve the initial value problem

$$\frac{y'(x)}{2x} - y(x) = 1, \quad y(1)=2.$$

b) Show that the solution is uniformly continuous on $[1,2]$ but not on $(1,\infty)$.

Solution.

a) The equation is well defined for $x \neq 0$, and especially close to $x=1$ (initial data). A particulate solution is $y_p(x)=-1$, but we must add a homogeneous solution y_h with $y_h(1)=3$ to satisfy the condition $y(1)=2$. As long as $x, y_h(x) \neq 0$, apply

$$\frac{y_h'(x)}{2x} - y_h(x) = 0 \iff \frac{y_h'(x)}{y_h(x)} = 2x \iff \ln|y_h(x)| = x^2 + C$$
$$\iff |y_h(x)| = e^C e^{x^2} \iff y_h(x) = \tilde{C} e^{x^2},$$

where C, $\tilde{C}$ are arbitrary constants. Now choose $\tilde{C}$ so that $y_h(1)=3$, that is, $\tilde{C}=\frac{3}{e}$. It follows that

$$y(x) = \frac{3}{e} e^{x^2} - 1,$$

which also solves the equation. (Note that y itself is defined for all $x \in \mathbb{R}$.)

It is possible to solve the equation e.g. also as a first-order linear equation by integrating factor or variation of the constant. Above, it is solved as a separable equation.

b)

Continuous functions are uniformly continuous on compact (closed and limited) interval. Since the above solution is continuous on $\mathbb{R}$, it therefore immediately follows that it is uniformly continuous on the interval $[1, 2]$.

Now consider $x_2 > x_1 > 1$. Since $x \mapsto y(x) = \frac{3}{e}e^{x^2} - 1$ is differentiable on $\mathbb{R}$, the mean value theorem gives

$$|y(x_1) - y(x_2)| = |y'(c)||x_1 - x_2| = \frac{6c}{e}e^{c^2}|x_1 - x_2| \geq e^{x_1^2}|x_1 - x_2|.$$

In other words: for a given $\varepsilon > 0$, no matter how small $|x_1 - x_2| < \delta$ is,

$$|y(x_1) - y(x_2)| > \varepsilon$$

if x_1 is large enough. The function is therefore not uniformly continuous on $(1, \infty)$.