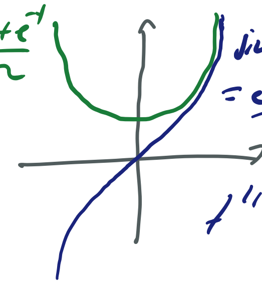
(iii) $\displaystyle\int_r^R \frac{dx}{\sqrt{ax+b}}$, $ax+b > 0$

$\cosh(t) = \frac{e^t+e^{-t}}{2}$

$f'' - f = 0$



$\sinh(t) = \frac{e^t - e^{-t}}{2}$

$f'' - f = 0$

$$= \begin{bmatrix} ax+b = u^2 \\ u\,du = a\,dx \end{bmatrix} = \frac{1}{a}\int_{\sqrt{ar+b}}^{\sqrt{aR+b}} \frac{u\,du}{u} = \frac{1}{a}\left(\sqrt{aR+b} - \sqrt{ar+b}\right).$$

$$u\frac{du}{dx} = a$$

Eks. august 2018

(1.) Vis ved induksjon at $\displaystyle\sum_{k=1}^{n} (2k-1) = n^2$ (*), $n \in \mathbb{N}$.

Basis $n=1$: $\displaystyle\sum_{k=1}^{1}(2k-1) = 2-1 = 1 = 1^2$. ok

(*) sann for $n=1$.

Antar at (*) sann for $n=N$. Vis for $n=N+1$.

$$\sum_{k=1}^{N+1}(2k-1) = \sum_{k=1}^{N}(2k-1) + \big(\underbrace{2(N+1)-1}_{2N+1}\big)$$

$$\underset{\substack{\text{(*) sann for } n=N \\ \text{(ind. antakelsen)}}}{=} N^2 + 2N + 1 = (N+1)^2 \quad \text{ok.}$$

(*) sann for $n = N+1$.

Induksjonsprinsippet $\implies$ (*) sann $\forall n \in \mathbb{N}$. $\blacksquare$

Alt. (uten induksjon)

$$\sum_{k=1}^{n} (2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = \cancel{2}\,\frac{n(n+1)}{\cancel{2}} - n$$

forelesn.

$$= n^2 + n - n = n^2. \quad \blacksquare$$

---

② Finn største og minste verdi til

$f(x) = 2x^3 + 3x^2 - 1$ på $[-2, 1]$.

Løsn.

- f kont (polynom)
- $[-2, 1]$ kompakt

ekstremalv. setn. $\implies$ $\exists$ maks/min $f(x)$, $x \in [-2, 1]$

$\exists f'(x)\ \forall x \in \mathbb{R} \implies$ maks/min enten i $x = -2$, $x = 1$, eller der $\underbrace{f'(x) = 0}_{\text{kritiske punkter}}$ for $-2 < x < 1$.

$f(-2) = -16 + 12 - 1 = -5$

$f(1) = 4$

$$f'(x) = 6x^2 + 6x = 6x(x+1) = 0 \iff$$

$x = 0$ eller $x = -1$.

$$f(-1) = -2 + 3 - 1 = 0$$

$$f(0) = -1.$$

$$f(x) = 2x^3 + 3x^2 - 1$$

$$\Rightarrow \max_{x \in [-2,1]} f(x) = 4 \quad \text{og} \quad \min_{x \in [-2,1]} f(x) = -5. \qquad \square$$

---

③ Bruk delvis integrasjon til å løse det ubestemte integralet

$$\int x^2 \cos(x)\,dx$$

Løsn.

$$\int x^2 \cos(x)\,dx \underset{\text{delvis int.}}{=} x^2 \sin(x) - 2\int x \sin(x)\,dx$$

$$= x^2 \sin(x) + 2x\cos(x) - 2\int \cos(x)\,dx$$

$$= (x^2 - 2)\sin(x) + 2x\cos(x) + C, \qquad C \in \mathbb{R}. \qquad \square$$

---

④ Løs initialverdiproblemet

$$y' + y\sin(x) = \sin(x), \qquad y(0) = 0.$$

$y(x)$

Løsn. Dersom $y_h' + y_h \sin(x) = 0$, og (H)

$y_p' + y_p \sin(x) = \sin(x)$ (P)

gjelder $y' + y\sin(x) = \sin(x)$ for $y = y_h + y_p$.

Så løs (H) generelt, og (P) partikulært:

- $y_h' = -\sin(x)\, y_h \overset{y_h \neq 0}{\iff} \dfrac{y_h'}{y_h} = -\sin(x)$

$$\iff \frac{d}{dx} \ln|y_h| = \frac{d}{dx}\cos(x)$$

kjerneregel.

$$\iff e^{\ln|y_h|} = e^{\cos(x) + C} = \tilde{C} e^{\cos(x)}$$

e^C

$$\iff \boxed{y_h = \tilde{C} e^{\cos(x)}}, \quad \tilde{C} \in \mathbb{R}.$$

- $y_p = 1$ løser $y_p' + y_p \sin(x) = \sin(x)$.

Så $\underline{y(x) = \tilde{C} e^{\cos(x)} + 1}$ er den generelle løsn.

$$y(0)=0 \implies \tilde{c}e+1=0 \iff \tilde{c}=-\frac{1}{e}=-e^{-1}.$$

$$\implies \boxed{y(x)=1-e^{\cos(x)-1}}.$$

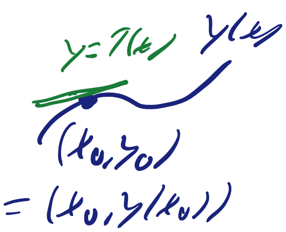
---

5 Bruk implisitt derivasjon til å finne tangenten til kurven

$$\boxed{x^3y^2-x^2y^3=4} \quad \text{i} \quad (x_0,y_0)=(2,1)$$

Løsn. Tangent i (x_0,y_0) der $y=y(x)$:

$$T(x)=y'(x_0)(x-x_0)+\underbrace{y_0}_{y(x_0)}$$



Må bestemme $y'(x_0)$ for $x_0=2$ $(y_0=1)$.

Gitt $y=y(x)$ deriverbar:

$$\frac{d}{dx}\left(x^3y^2-x^2y^3\right)\overset{\text{prod.reg.}}{\underset{\text{kjerneregel.}}{=}}\underline{3x^2y^2+2x^3yy'}$$

$$\underline{-2xy^3-3x^2y^2y'}=\underbrace{0}_{\frac{d}{dx}4}$$

$$\underline{x_0=2,\ y_0=1} \implies$$

$12 + 16y' - 4 - 12y' = 0$

$\Leftrightarrow 8 + 4y' = 0 \Leftrightarrow y' = -2$ i punktet $(x_0, y_0) = (2, 1)$.

Så $\boxed{y = -2(x-2) + 1}$ også feil i ltt.

gir tangenten

---

6 Løs det ubestemte integralet

$$\int \frac{2x^2\,dx}{(x^2+4)(x+2)} \quad (*)$$

Løsn. $x+2$ enkel | $\deg(x^2) < \deg((x^2+4)(x+2))$

x^2+4 irredusibel | (ellers polynomdivisjon)

Ansatz: $\dfrac{2x^2}{(x^2+4)(x+2)} = \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x+2}$.

- $2x^2 = (Ax+B)(x+2) + C(x^2+4)$, $x \neq -2$

$x \to -2 \Rightarrow 8 = 8C \Leftrightarrow C = 1$

- $2x^2 = Ax^2 + (2A+B)x + 2B + Cx^2 + 4C$

$$\iff \begin{cases} A = C = 1 \\ B = -2 \end{cases}$$

Så $(*) = \int \frac{x-2}{x^2+4}\,dx + \int \frac{dx}{x+2}$

$$= \underbrace{\int \frac{x}{x^2+4}\,dx}_{\frac{1}{2}\ln|x^2+4| + C} - 2\int \frac{dx}{x^2+4} + \underbrace{\int \frac{dx}{x+2}}_{\ln|x+2| + C}$$

$$\frac{2}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} = \begin{bmatrix} u = \frac{x}{2} \\ du = \frac{1}{2}\,dx \end{bmatrix} = \cancel{\frac{2\cdot 2}{4}}\int \frac{du}{u^2+1}$$

$$= \arctan(u) + C = \arctan\left(\frac{x}{2}\right) + C.$$

$$\implies (*) = \frac{1}{2}\ln|x^2+4| + \ln|x+2| - \arctan\left(\frac{x}{2}\right) + C,\quad C \in \mathbb{R}.$$

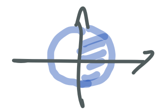
---

7 (i) Vis at $f(x) = \int_0^{\sin(x)} e^{\arcsin(t)}\,dt$ er veldefinert for $x \in \mathbb{R}$.

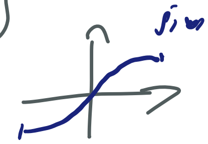
(ii) Beregn $f'(x)$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Løsn. - $\sin : \mathbb{R} \to [-1, 1]$ kont. og veldefinert.

- $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ kont og bijektiv, med deriverte $\sin'(x) \neq 0$ på $(-\frac{\pi}{2}, \frac{\pi}{2})$.



- invers $\arcsin : [-1, 1] \xrightarrow{\text{kont}} [-\frac{\pi}{2}, \frac{\pi}{2}]$ (deriverbar på $(-1, 1)$)



(i) $\begin{cases} t \mapsto e^{\arcsin(t)} \text{ kont på } [-1,1]. \\ \sin(x) \in [-1,1] \text{ for hvert } x \in \mathbb{R}. \end{cases}$

$\Longrightarrow$ integralet $\exists$ med endelig verdi

(ii) Kjedjereg. + fund. setn. $\Longrightarrow$

periodisk

$$\frac{d}{dx} \int_0^{\sin(x)} e^{\arcsin(t)}\, dt = \boxed{\underline{e^{\arcsin(\sin(x))}} \cdot \sin'(x) \quad \forall x \in \mathbb{R}}$$

$$= e^x \cos(x) \quad \forall x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

$\uparrow$

$\forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

---

(8) f kont på $[a,b]$ og deriverbar på (a,b).

Vis: f konstant på $[a,b]$ $\Longleftrightarrow$ $f'(x) = 0 \quad \forall x \in (a,b)$.

Løsn. "$\Leftarrow$" Antakelser + middelv. setn.

$\exists c \in (a,b): \quad f(x)-f(y) = \underbrace{f'(c)}_{=0 \text{ enl. forutsetn.}}(x-y) \quad \forall x,y \in [a,b].$

$\Rightarrow \boxed{f(x)=f(y) \ \forall x,y \in [a,b]}$

Fikser y (f.eks. $y=a$) $\Rightarrow f(x)=f(a) \ \forall x \in [a,b]$.

"$\Rightarrow$" f konst. $\Rightarrow \dfrac{f(x+h)-f(x)}{h} = 0 \quad \forall x, x+h \in (a,b)$

$\underset{\text{def } f'}{\Rightarrow} \quad f'(x)=0 \quad \forall x \in (a,b).$ #

---

9 La $a \in \mathbb{R}$ og betrakt

$$g_a(x) = \begin{cases} x|x|^a, & x \neq 0 \\ 0, & x=0. \end{cases}$$

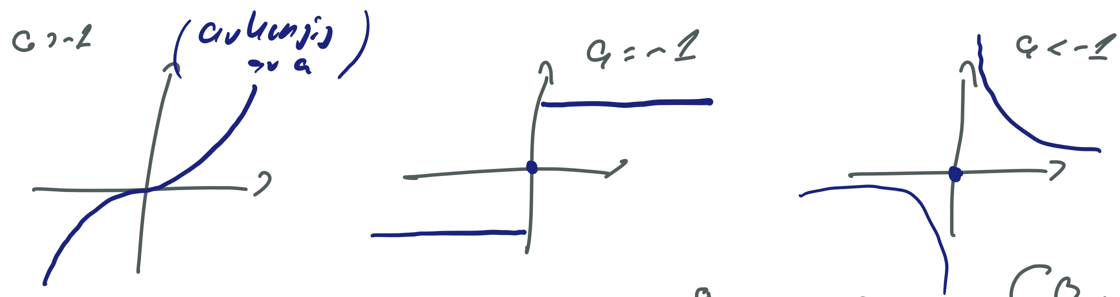
(a) For hvilke a er g_a kont. i $x=0$?

(b) — " — deriverbar i $x=0$?

Løsn. (a) Kont. i $x_0=0$?

$|g_a(x)-g_a(0)| \underset{x\neq 0}{=} |x|x|^a - 0| = |x|^{a+1} \xrightarrow[x\to 0]{} 0$

$\Leftrightarrow a+1>0 \Leftrightarrow \boxed{a>-1.}$



b) $\dfrac{g_a(0+h) - g_a(0)}{h} = \dfrac{h|h|^a}{h} = |h|^a \longrightarrow \begin{cases} 0, & a>0 \\ 1, & a=0 \\ \infty, & a<0. \end{cases}$

Så $\exists\, g'(0)$ nøyaktig når $a \geq 0$.

Numerisk integrasjon

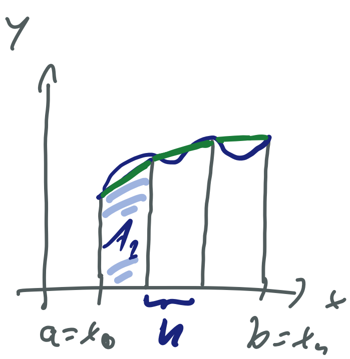
① Trapesmetoden

La $f \in C^2([a,b], \mathbb{R})$.

Partisjon $x_j = a + jh$, $j = 0, 1, \ldots, n$;

$h = \dfrac{b-a}{n} \Longrightarrow x_0 = a$, $x_n = b$ og

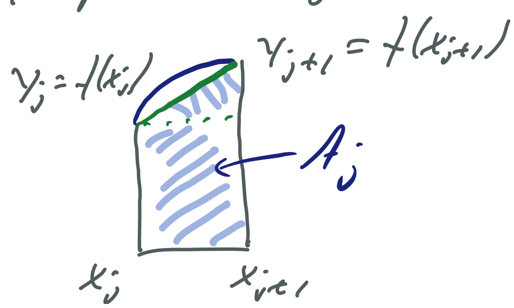
$x_{j+1} - x_j = h \quad \forall\, j = 0, \ldots, n-1.$



Approksimer arealet med trapesoider A_j

Areal $A_j = h y_j + \dfrac{h(y_{j+1} - y_j)}{2}$

$= \dfrac{h(y_j + y_{j+1})}{2}.$



Summer over $j \Rightarrow \sum_{j=0}^{n-1} h\frac{(y_{j+1}+y_j)}{2}$

$y_j = f(x_j)$

$$= \boxed{h\left[\frac{y_0}{2} + y_1 + \dots + y_{n-1} + \frac{y_n}{2}\right]}$$

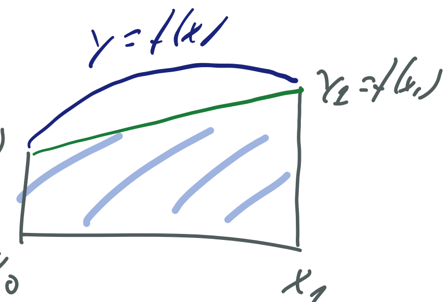
Def. n'te trapesapproksimasjonen, T_n, til $\int_a^b f(x)\,dx$.

Teorem $f \in C^2([a,b], \mathbb{R})$

$$\Rightarrow \left| \int_a^b f(x)\,dx - T_n \right| \leq \frac{(b-a)^3}{12\, n^2} \max_{[a,b]} |f''|$$

Merk: feilen $\to 0$ kvadratisk; $h = \frac{b-a}{n}$.

Bevis Betrakt $[x_0, x_1]$.



La $g(x) = f(x) - \underbrace{\left[f(x_0) + \frac{x - x_0}{x_1 - x_0}\left(f(x_1) - f(x_0)\right)\right]}_{\text{affin funksjon}}$

$g(x_0) = f(x_0) - f(x_0) = 0.$

$g(x_1) = f(x_1) - f(x_1) = 0.$

$A + Bx \Rightarrow g''(x) = f''(x)$

På ene siden: $\displaystyle\int_{x_0}^{x_1} g(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx - f(x_0)(x_1 - x_0)$

$$\underbrace{-\frac{(x-x_0)^2}{2(x_1-x_0)}\Bigg|_{x_0}^{x_1}\big(f(x_1)-f(x_0)\big)}_{-\frac{1}{2}(x_1-x_0)(f(x_1)-f(x_0))} = \int_{x_0}^{x_1} f(x)\,dx - \frac{x_1-x_0}{2}\big(f(x_0)+f(x_1)\big)$$

$$\boxed{= \int_{x_0}^{x_1} f(x)\,dx - \frac{h}{2}(y_0 + y_1).}$$

Men også: $\displaystyle\int_{x_0}^{x_1} 1\cdot g(x)\,dx \underset{\text{int.}}{\overset{\text{delvis}}{=}} \underbrace{\left(x - \frac{x_1-x_0}{2}\right)g(x)\Bigg|_{x_0}^{x_1}}_{=0}$

$\frac{d}{dx} \rightsquigarrow 1$

$$-\int_{x_0}^{x_1}\left(x - \frac{x_1-x_0}{2}\right)g'(x)\,dx \underset{\text{int.}}{\overset{\text{delvis}}{=}} \cancel{\frac{1}{2}(x-x_0)(x_1-x)\,g'(x)\Bigg|_{x_0}^{x_1}}$$

$\frac{d}{dx} \rightsquigarrow -\left(x - \frac{x_1-x_0}{2}\right)$

$$-\frac{1}{2}\int_{x_0}^{x_1}(x-x_0)(x_1-x)\underbrace{g''(x)}_{f''(x)}\,dx.$$

Så feilen $\displaystyle\left|\int_{x_0}^{x_1} g(x)\,dx\right| \underset{\Delta\text{-ulikh.}}{\le} \frac{1}{2}\int_{x_0}^{x_1}(x-x_0)(x_1-x)|f''(x)|\,dx$

$$\le \frac{1}{2} \max_{[x_0,x_2]} |f''| \int_{x_0}^{x_1} (x-x_0)(x_1-x)\,dx = \frac{1}{12}(x_1-x_0)^3 \max |f''|$$

regning

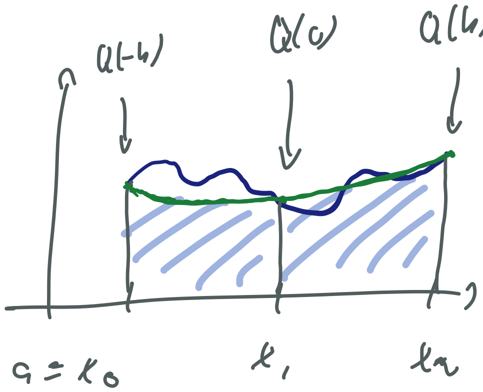
Likt for intervaller $[x_j, x_{j+1}] \Longrightarrow$ $\quad x_{j+1}-x_j = \frac{b-a}{n}$.

$$\left| \int_a^b f(x)\,dx - T_n \right| \le \sum_{j=0}^{n-1} \max_{[x_j,x_{j+1}]} |f''| \frac{h^3}{12}$$

$$\le \frac{(b-a)^3}{12n^2} \max_{[a,b]} |f''| .$$

Merk: Approksimasjon til lineær orden gir kvadratisk feil (typisk).

② Simpsons metode – kvadratisk approksimasjon (parabel)



$Q(t) = A + Bt + Ct^2$

$x_0 = x_1 - h$

$x_2 = x_1 + h$

Ønsker: $Q(-h) = f(x_0)$, $Q(0) = f(x_1)$, $Q(h) = f(x_2)$.

$$\begin{cases} A - Bh + Ch^2 = f(x_0) \\ A = f(x_1) \\ A + Bh + Ch^2 = f(x_2) \end{cases} \Longrightarrow \begin{cases} A = f(x_1) \\ 2Ch^2 = f(x_0) + f(x_2) - 2f(x_1) \end{cases}$$

like

$$\int_{-h}^{h} (A + \cancel{Bt} + Ct^2)\,dt = 2\int_0^h (A + Ct^2)\,dt$$

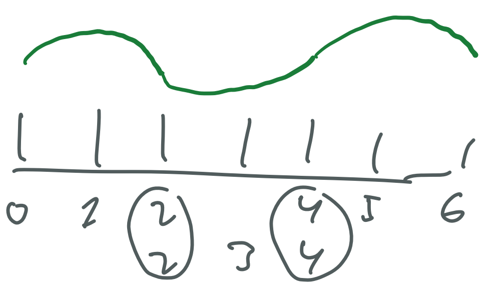
odde!

$$= 2\left[At + \frac{Ct^3}{3}\right]_0^h = 2h\left[A + \frac{C}{3}h^2\right]$$

$$= h\left[\frac{6}{3}f(x_1) + \frac{f(x_0) - 2f(x_1) + f(x_2)}{3}\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

Summerer over j $\Longrightarrow$ $\int_a^b f(x)\,dx \approx S_n$,



$$\boxed{S_n \overset{\text{def}}{=} \frac{h}{3}\left[f(x_0) + 4\sum_{j \text{ odde}}^{j \neq n} f(x_j) + 2\sum_{j \text{ partall}}^{j \neq 0,n} f(x_j) + f(x_n)\right]}$$

Teorem (Simpsons regel)

$f \in C^4([a,b], \mathbb{R})$

$$\Longrightarrow \left| \int_a^b f(x)\,dx - S_n \right| \le \frac{(b-a)^5}{n^4} \max_{[a,b]} |f^{(4)}|$$

Generelt: 'splines' – polynomer av grad N.

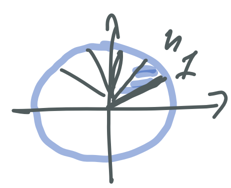
Eks. Oppg. 5 20/12 2017

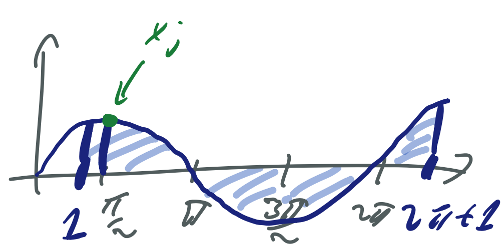
(a) Finn grenseverdien $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} \frac{2\pi}{n} \sin\left(1+\frac{2\pi j}{n}\right)$.

Løsn. Riemannsum med $\frac{2\pi}{n} = \frac{b-a}{n} = h$.

$(x_0 = 1)$, $x_1 = 1 + \frac{2\pi}{n}$, $\ldots$, $x_n = 1 + 2\pi$

$\sin\left(1 + \frac{2\pi j}{n}\right) = \sin(x_j)$





2π

$$\text{Så } \lim_{n\to\infty} \sum_{j=1}^{n} \frac{2\pi}{n} \sin\left(1+\frac{2\pi j}{n}\right) = \int_0^{2\pi} \sin(1+x)\,dx$$

sin integrerbar

$$= -\cos(1+x)\Big|_0^{2\pi} = 0.$$

cos 2π-per.

(b) Vis ved hjelp av ε/δ at $\lim_{x\to 1}(3x+1) = 4$.

Løsn. La $\varepsilon > 0$, $f(x) = 3x+1$, $L = 4$, $x_0 = 1$.

$$|f(x) - L| = |3x+1-4| = |3x-3| = 3|x-1| \overset{!}{<} \varepsilon.$$

x_0

dersom $|x - x_0| = |x-1| < \delta \leq \frac{\varepsilon}{3}$.

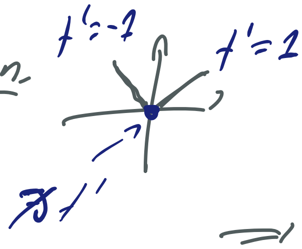
Velg $\delta = \frac{\varepsilon}{3} \Rightarrow |3x+1-4| < \varepsilon$ når $|x-1| < \delta$. $\blacksquare$

Eks (Oppg. 7 fra samme eks!)

Vis ved et moteksempel at følgende er usann:

(a) $f \in C([a,b], \mathbb{R}) \Rightarrow \exists\, c \in (a,b);\ f'(c) = \frac{f(b)-f(a)}{b-a}$

Løsn.



La $f(x) = |x|$ kont.!

$a = -1,\ b = 1.$

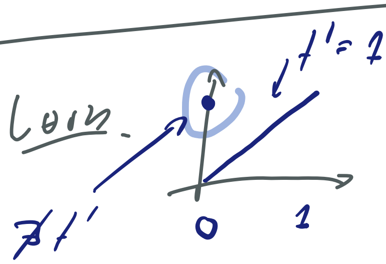
$$\Rightarrow \frac{f(1)-f(-1)}{1-(-1)} = \frac{|1|-|-1|}{2} = 0$$

men $\nexists\, c \in (-1,1);\ f'(c) = 0.$ ↯

(b) $f : [a,b] \to \mathbb{R}$ deriverbar på (a,b)

$\Rightarrow \exists\, c \in (a,b);\ f'(c) = \frac{f(b)-f(a)}{b-a}$ (altså usant!)

Løsn.



Velg $f(x) = \begin{cases} x, & x > 0 \\ 1, & x = 0. \end{cases}$

def. på $[0,1]$

og deriverbar på $(0,1)$.

$$\Rightarrow \frac{f(1)-f(0)}{1-0} = \frac{1-1}{1} = 0$$ men $\nexists\, c \in (0,1);$

$f'(c) = 0.$
