høyest indeks.

La $n_1 = N+1 \Rightarrow x_{n_1}$ ikke en topp

$\Rightarrow \exists\, n_2 > n_1 ;\ x_{n_2} \geq x_{n_1}$.

Nå: x_{n_2} ikke en topp

$\Rightarrow \exists\, n_3 > n_2 ;\ x_{n_3} \geq x_{n_2}$.

...

$\Rightarrow \exists$ monoton delfølge $(x_{n_i})_i$;

$x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \dots$

I begge tilfellene (i) og (ii) har vi en monoton delfølge. ⊠

Lemma 2: En kontinuerlig funksjon på et kompakt intervall er begrenset.

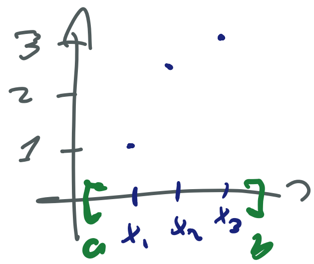
Bevis Motsielsebevis: anta at f ikke er begrenset ovenifra.

$\Rightarrow \forall B \geq 1 \ \exists\, x_B \in [a,b] : f(x_B) \geq B$

Velg $B = 1, 2, 3, \dots$

$\Longrightarrow \forall n \geq 1 \ \exists x_n \in [a,b] : f(x_n) \geq n.$

Da er følgen $(x_n)_n \subset [a,b]$ begrenset.



Lemma 1 $\Longrightarrow$ $\exists$ monoton og begrenset delfølge $(x_{n_j})_j$, si voksende.

kompl. aksiom. $\Longrightarrow$ $\lim_{j \to \infty} x_{n_j} = \sup\{x_{n_j}\} \overset{\text{def.}}{=} x_0 \in [a,b].$

obs! $x_{n_j} \leq b$ og $|x_{n_j} - x_0| \to 0 \Rightarrow x_0 \leq b$

Nå: f kont på $[a,b]$ $\Longrightarrow$ $\boxed{f(x_{n_j}) \to f(x_0) \text{ da } x_{n_j} \to x_0.}$

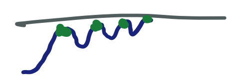
Men dette er en motsigelse!

$$\underbrace{n_j}_{\to \infty} - |f(x_0)| \leq \underbrace{|f(x_{n_j})|}_{\to \infty} - \underbrace{|f(x_0)|}_{\text{fiksert}} \overset{\text{D-ulik.}}{\leq} |f(x_{n_j}) - f(x_0)| \longrightarrow 0$$

Altså: f begrenset ovenfra.

Bevis for ekstremalverdisetningen

Lemma 2 $\Rightarrow \sup\limits_{x\in[a,b]} f(x) = M \in \mathbb{R}$

M 

Supremum minste øvre begrensning

$\Rightarrow \forall n \geq 1 \ \exists x_n \in [a,b] : 0 \le M - f(x_n) < \frac{1}{n}$,

dvs $f(x_n) \to M$ da $n \to \infty$

Velg en monoton delfølge $(x_{n_j})_j \subset [a,b]$

$\mathbb{R}$ komplett $\Rightarrow \exists x_0 = \lim\limits_{j\to\infty} x_{n_j}$ og $x_0 \in [a,b]$.

f kont $\Rightarrow f(x_{n_j}) \to f(x_0)$ da $j \to \infty$

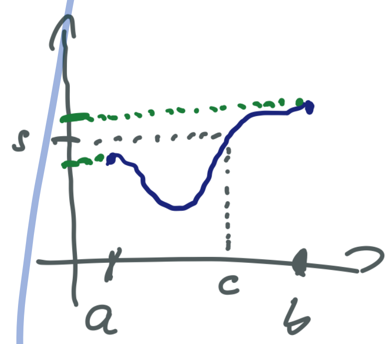
Så: $|f(x_0) - M| \overset{\Delta\text{-ulik.}}{\le} \underbrace{|f(x_0) - f(x_{n_j})|}_{\to 0} + \underbrace{|f(x_{n_j}) - M|}_{\to 0}$,

dvs har funnet $\boxed{x_0 \in [a,b] ; \ f(x_0) = \sup\limits_{x\in[a,b]} f(x)}$

Teorem: Skjæringssetningen

La $f \in C([a,b],\mathbb{R})$.

For hvert s liggende mellom $f(a)$ og $f(b)$, eksisterer $c \in (a,b)$; $\boxed{f(c)=s}$



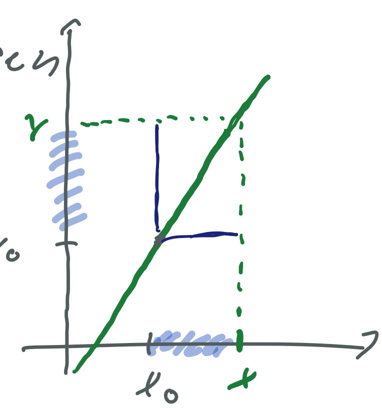
Korollar: En kont. funksjon på et kompakt intervall antar alle verdier mellom sitt maks og min på samme intervall.

Deriverte

En rett linje har likningen

$$\boxed{\frac{y-y_0}{x-x_0}=m}$$

retnings-koeffisient

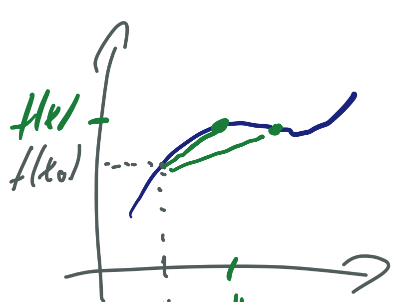


$x \neq x_0$ $\Longleftrightarrow$ $\boxed{y = \underbrace{y_0}_{\text{konst.}} + \underbrace{m(x - x_0)}_{\text{lineær i } x}}$

- En lineær funksjon $f : \mathbb{R} \to \mathbb{R}$ er gitt ved $\boxed{f(x) = f(x_0) + m(x - x_0)}$

- For en ikke-lineær funksjon er

$\dfrac{f(x) - f(x_0)}{x - x_0} \neq$ konst., men et lokalt mål på retningen til grafen til f.



$\boxed{\text{Vi ønsker å la } x \to x_0}$

Def. • $f : I \overset{\text{åpent}}{\subset} \mathbb{R} \to \mathbb{R}$, $x_0 \in I$

er <u>deriverbar i x_0</u> $\overset{\text{def}}{\iff}$

$$\exists \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad (\text{i } \mathbb{R}).$$

Vi skriver da $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$,

alt. $\frac{df}{dx}(x_0)$, $Df(x_0)$, ...

- f deriverbar på I $\overset{\text{def.}}{\iff}$ $\exists f'(x) \ \forall x \in I$.

  <u>Funksjonen</u> $f' : I \to \mathbb{R}$, $x \mapsto f'(x)$,

  kalles <u>den deriverte til f</u>.

<u>Teorem</u>: En deriverbar funksjon er kontinuerlig.

<u>Bevis</u> La $x_0 \in I$, $f : I \to \mathbb{R}$ deriverbar.

<u>Vet</u>: $\frac{f(x) - f(x_0)}{x - x_0} \to f'(x_0)$ da $x \to x_0$.

<u>Vil vise</u>: $f(x) \to f(x_0)$ da $x \to x_0$.

$$|f(x) - f(x_0)| \overset{x \neq x_0}{=} \left| \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \right|$$

$$= \underbrace{\left|\frac{f(x)-f(x_0)}{x-x_0}\right|}_{\to f'(x_0)} \underbrace{|x-x_0|}_{\to 0} \to f'(x_0)\cdot 0 = 0 \quad \text{da } x \to x_0,$$

prod av endelige grenser.

dvs $f(x) \to f(x_0)$.

Eks. $f : \mathbb{R} \to \mathbb{R}, \ x \mapsto x^2$ deriverbar

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{x^2-x_0^2}{x-x_0} = \frac{\cancel{(x-x_0)}(x+x_0)}{\cancel{x-x_0}}$$

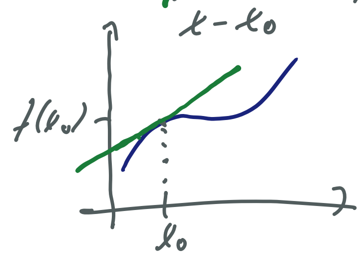
$$\underset{x \neq x_0}{=} x + x_0 \to 2x_0 \quad \text{da } x \to x_0$$

Så $f' : \mathbb{R} \to \mathbb{R}$ er funksjonen $x \mapsto 2x$.

Linjen $\boxed{y = f(x_0) + f'(x_0)(x-x_0)}$

kalles tangenten til f i x_0

$$\frac{y - f(x_0)}{x-x_0} = f'(x_0)$$



Prop. Derivasjon er en lineær operasjon, dvs

$$\boxed{(\lambda f + \mu g)' = \lambda f' + \mu g'}$$

$\forall \lambda, \mu \in \mathbb{R}$ og f, g deriverbare.

Bevis: le boken.

Prop. Produktregelen: $\boxed{(fg)' = f'g + fg'}$

dersom f', g' eksisterer.

Merk at $\dfrac{f(x) - f(x_0)}{x - x_0} \underset{h = x - x_0}{=} \dfrac{f(x_0 + h) - f(x_0)}{h}$

Bevis $\dfrac{(fg)(x_0 + h) - (fg)(x_0)}{h}$

$$= \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h}$$

$$= \left(\frac{f(x_0+h) - f(x_0)}{h}\right) \underset{\to g(x_0)}{g(x_0+h)} + f(x_0)\left(\frac{g(x_0+h) - g(x_0)}{h}\right)$$

$\longrightarrow \underline{f'(x_0)\cdot g(x_0) + f(x_0)g'(x_0)}$ da $h \to 0$. $\square$

Prop. $\boxed{\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}}$ dersom $g \neq 0$ og g' eksisterer.

Bevis $\frac{1}{h}\left(\frac{1}{g(x_0+h)} - \frac{1}{g(x_0)}\right) = \frac{1}{h}\left(\frac{g(x_0)-g(x_0+h)}{g(x_0+h)g(x_0)}\right)$

$$= -\frac{1}{\underbrace{g(x_0+h)}_{\to g(x_0)}\, g(x_0)} \underbrace{\left[\frac{g(x_0+h)-g(x_0)}{h}\right]}_{\to g'(x_0)} \longrightarrow -\frac{g'(x_0)}{(g(x_0))^2}. \quad \square$$

Korollar: $\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$, dersom $\exists f', g'$ og $g \neq 0$.

Bevis $\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' \underset{\text{prod.reg.}}{=} f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'$

$$\underset{\left(\frac{1}{g}\right)'}{=} f' \cdot \frac{1}{g} + f\left(-\frac{g'}{g^2}\right) = \frac{f'g - fg'}{g^2}. \quad \square$$

Teorem: kjerneregelen

$$(f\circ g)' = (f'\circ g)g',$$ dersom høyre-leddet er veldefinert.

dvs $$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$ (uten bevis). $\square$

Eks. $f(y) = e^y, \quad f'(y) = e^y$

$g(x) = x^3, \quad g'(x) = 3x^2.$

$f\circ g : \mathbb{R} \to \mathbb{R}, \quad x \longmapsto e^{x^3}$

$$(f\circ g)'(x) = (f'\circ g)(x)\cdot g'(x)$$
$$= f'(g(x))\, g'(x) = 3x^2 e^{x^3}.$$

$\square$

Prop. Derivasjon er en lineær operasjon

$$(\lambda f + \mu g)' = \lambda f' + \mu g',$$

$\lambda, \mu \in \mathbb{R}$, f, g deriverbare.

$x_0 + h = x \iff x - x_0 = h$

Bevis

$$(\lambda f+\mu g)'(x_0) = \lim_{h\to 0} \frac{(\lambda f+\mu g)(x_0+h)-(\lambda f+\mu g)(x_0)}{h}$$

$$\overset{\text{def}}{=} \lim_{h\to 0}\left[\frac{\lambda(f(x_0+h)-f(x_0))}{h} + \frac{\mu(g(x_0+h)-g(x_0))}{h}\right]$$

$$\underset{\exists f', g'}{\overset{\text{sum grenser}}{=}} \lambda \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} + \mu \lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}$$

$$= \lambda f'(x_0) + \mu g'(x_0).$$

Noen viktige deriverte

- $\dfrac{d}{dx} x^n = n x^{n-1}, \quad n = 1, 2, 3, \ldots$

$\boxed{n=1}$ $\quad \dfrac{(\cancel{x}+h)-\cancel{x}}{h} = \dfrac{h}{h} = 1 \longrightarrow 1$ da $h\to 0$

så $\dfrac{d}{dx} x = 1.$

$\boxed{n=2}$ ✓ se eksempel

$\boxed{n=3}$ $\quad \dfrac{(x+h)^3 - x^3}{h} = \dfrac{\cancel{x^3} + \overset{\sim h}{3x^2\cancel{h}} + \overset{\sim h^2}{3xh^2} + \overset{\sim h^3}{h^3} - \cancel{x^3}}{\cancel{h}}$

$$= 3x^2 + \underbrace{h(3x+h)}_{B(h) \text{ begrenset da } h\to 0} \longrightarrow 3x^2 \text{ da } h \to 0.$$

$\boxed{\text{Generell } n}$ $\dfrac{(x+h)^n - x^n}{h}$

$$= \frac{\cancel{x^n} + nhx^{n-1} + h^2 B(h) - \cancel{x^n}}{h}$$

$$= nx^{n-1} + hB(h) \longrightarrow nx^{n-1} \text{ da } h \to 0.$$

$\square$

- $\dfrac{d}{dx} x^{-n} = \dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) \underset{\left(\frac{1}{g}\right)'}{=} -\dfrac{(x^n)'}{(x^n)^2}$

$$= -\frac{nx^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}}, \quad n \in \mathbb{N}.$$

- Generelt: $\dfrac{d}{dx} x^r \overset{\text{uten bevis}}{=} r x^{r-1}, \; r \in \mathbb{R}, \; x \neq 0.$

Eks. $\dfrac{d}{dx}\sqrt{x} = \dfrac{d}{dx} x^{\frac{1}{2}} = \dfrac{1}{2} x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}.$

- Obs! også at $(e^x)' = e^x.$

Def. Vi sier at f er <u>2 ggr deriverbar</u>

dersom $\exists (f')' \overset{\text{def}}{=} f''$.

Skriver $f^{(2)}$.

Eks. $\left(\frac{d}{dx}\right)^2 \sqrt{x} = \frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{4}\frac{1}{x^{3/2}}$.

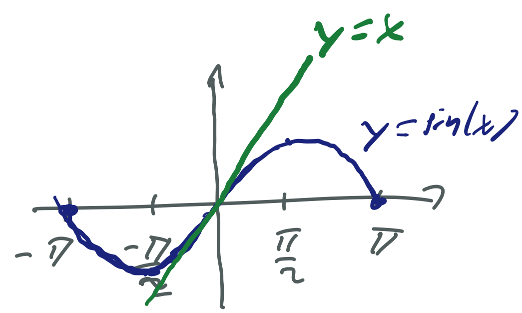
<u>n ggr deriverbar</u>: $\exists \left(\frac{d}{dx}\right)^n f \overset{\text{def}}{=} f^{(n)}$.

Deriverte til (noen) trig funksj.

Uten bevis (finnes geom. arg. i AE).

- sin og cos: $\mathbb{R} \to [-1,1]$ er kont.
- $|\sin(x)| \leq |x| \quad \forall x \in \mathbb{R}$

og $\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1.}$



Dertra får vi:

(i) $\boxed{\frac{\cos(x)-1}{x}} \to 0$ da $x \to 0$:

$$\sin^2(\alpha) = \frac{\boxed{1-\cos(2\alpha)}}{2}, \quad \text{velg } \alpha = \frac{x}{2}.$$

$$\Longrightarrow \frac{\cos(x)-1}{x} = -\frac{2\sin^2(\frac{x}{2})}{x}$$

$$= -\sin\left(\tfrac{x}{2}\right)\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} = -\underbrace{\sin(\alpha)}_{\to 0}\underbrace{\frac{\sin(\alpha)}{\alpha}}_{\to 1}$$

$$\longrightarrow -0\cdot 1 = 0 \quad \text{da } \alpha \to 0 \text{ da } x \to 0.$$

(ii) $\sin' = \cos$:

$$\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \sin(x)\underbrace{\frac{[\cos(h)-1]}{h}}_{\to 0} + \cos(x)\underbrace{\frac{\sin(h)}{h}}_{\to 1}$$

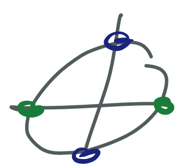
$$\longrightarrow \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x) \quad \text{da } h \to 0.$$

$\#$

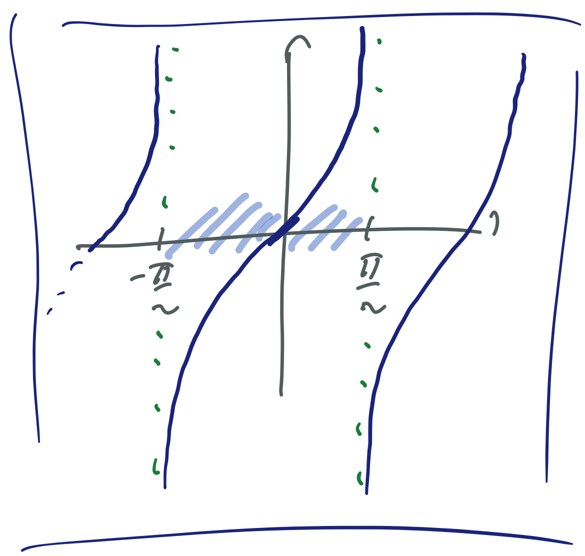
(iii) $\cos' = -\sin$ (øving)

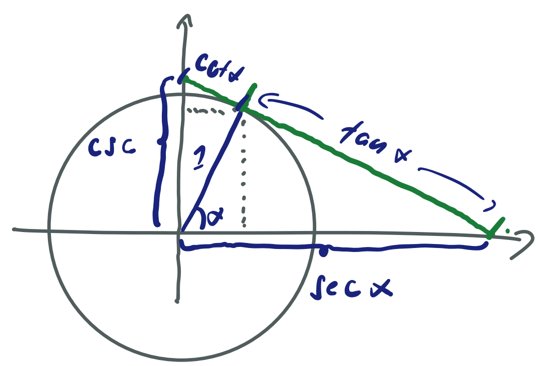
Fler trig. funksj.

- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}, \quad x \neq \dfrac{\pi}{2} + k\pi, \; k \in \mathbb{Z}.$



- $\text{cotan}(x) = \dfrac{\cos(x)}{\sin(x)}, \quad x \neq k\pi, \; k \in \mathbb{Z}.$





$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$\overset{(f/g)'}{=} \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{(\cos(x))^2}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \overset{\text{def.}}{=} \sec^2(x).$$

$$\boxed{\csc(x) = \frac{1}{\sin(x)}}$$

Eks grenseverdier

- Finn $\displaystyle\lim_{x\to\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$.

Løsning: Da $x\to\infty$ kan vi betrakte $x \geq 1$.

Da er $$\frac{2x-1}{\sqrt{3x^2+x+1}} = \cancel{\frac{x}{\sqrt{x^2}}}\, \frac{2-\frac{1}{x}}{\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}$$

$$\underset{x>0}{=} 1\cdot \frac{2-\underbrace{\frac{1}{x}}_{\to 0}}{\sqrt{3+\underbrace{\frac{1}{x}}_{\to 0}+\underbrace{\frac{1}{x^2}}_{\to 0}}} \longrightarrow 1\cdot \frac{2-0}{\sqrt{3+0+0}} = \frac{2}{\sqrt{3}}.$$

- Finn $\displaystyle\lim_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

$$a-b = \frac{a^2-b^2}{a+b}$$

$$\sqrt{x^2+2x}-\sqrt{x^2-2x}$$

$$= \frac{|x^2+2x|-|x^2-2x|}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$$

$$\overset{x\to-\infty}{\underset{x^2\pm2x>0}{=}} \frac{\cancel{x^2}+2x-(\cancel{x^2}-2x)}{\sqrt{+}+\sqrt{-}} = \frac{4x}{\sqrt{\ }+\sqrt{\ }}$$

$$= \frac{\cancel{x}(-1)}{\cancel{\sqrt{x^2}}} \cdot \frac{4}{\sqrt{1+\underbrace{\frac{2}{x}}_{\to 0}} + \sqrt{1-\underbrace{\frac{2}{x}}_{\to 0}}} \longrightarrow \frac{-1 \cdot 4}{\sqrt{1+0}+\sqrt{1-0}}$$

$x < 0$

$= -2$ da $x \to -\infty$.

$\not\exists$