For any $p \geq 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the <u>harmonic series of order p</u>.

We will show that for $p=1$ the harmonic series diverges.

<u>LEMMA 4.7</u>: If a sequence $(x_n)_{n=1}^{\infty}$ converges to $l \in \mathbb{R}$, then every subsequence $(x_{k_n})_{n=1}^{\infty}$ also converges to $l \in \mathbb{R}$.

<u>PROOF</u>

Let $(x_{k_n})_{n=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$.

Let $\varepsilon > 0$. Because $\lim_{n\to\infty} x_n = l$, there exists $n_0 \in \mathbb{N}$ such that

$$|x_n - l| < \varepsilon \quad \text{for all } n \geq n_0.$$

Since $k_n \geq n$ for any $n = 1, 2, \dots$ therefore for all $n \geq n_0$, $k_n \geq n \geq n_0$ and thus $|x_{k_n} - l| < \varepsilon$ for all $n \geq n_0$. ∎

THEOREM 4.8: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

PROOF

Let $(S_n)_{n=1}^{\infty}$ be the sequence of partial sums, i.e.

$$S_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}, \quad n = 1, 2, \dots$$

By Lemma 4.7 it suffices to find a subsequence of $(S_n)_{n=1}^{\infty}$ which diverges.

$$\begin{aligned}
S_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n} \\
&= 1 + \frac{1}{2^1} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\
&\qquad \dots + \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \dots + \frac{1}{2^n}\right) \\
&> 1 + \frac{1}{2} + \left(\overbrace{\frac{1}{4} + \frac{1}{4}}^{\#2}\right) + \left(\overbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}^{\#4}\right) + \\
&\qquad \dots + \left(\underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \dots + \frac{1}{2^n}}_{\# 2^{n-1}}\right) \\
&= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \dots + \frac{2^{n-1}}{2^n} \\
&= 1 + \frac{n}{2} \longrightarrow \infty \quad (\text{as } n \to \infty). \quad \blacksquare
\end{aligned}$$

We say that $\sum_{n=1}^{\infty} a_n$ is a <u>telescopic</u> series if $a_n = b_{n+1} - b_n$ for some sequence $(b_n)_{n=1}^{\infty}$.

If $\sum_{n=1}^{\infty} a_n$ is a telescopic series with $a_n = b_{n+1} - b_n$, $n = 1, 2, \ldots$ and $\lim_{n\to\infty} b_n = l \in \mathbb{R}$ then $\sum_{n=1}^{\infty} a_n$ converges.

Indeed, if $S_n = a_1 + a_2 + \ldots + a_n$ then

$$\begin{aligned} S_n &= a_1 + a_2 + \ldots + a_n \\ &= (\cancel{b_2} - b_1) + (\cancel{b_3} - \cancel{b_2}) + \cancel{\ldots} + (b_{n+1} - \cancel{b_n}) \\ &= b_{n+1} - b_1 \\ &\to l - b_1 . \end{aligned}$$

E.g. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and find its value.

- $\frac{1}{n(n+1)} = \frac{n+1-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

The N-th partial sum of the series is

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\ldots} + \cancel{\tfrac{1}{N}} - \frac{1}{N+1} \\ &= 1 - \frac{1}{N+1} \to 1 \text{ as } N \to \infty . \end{aligned}$$

We have proved that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

In the specific example of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ it was easy to spot that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

We could have argued as follows:
We want to find coefficients $A, B \in \mathbb{R}$ such that

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \forall x \in \mathbb{R} \setminus \{0,-1\} \Rightarrow$$

$$A(x+1) + Bx = 1 \quad \forall x \in \mathbb{R} \setminus \{0,-1\} \Rightarrow$$

$$(A+B)x + (A-1) = 0 \quad \forall x \in \mathbb{R} \setminus \{0,-1\}$$

So

$$\left.\begin{array}{l} A - 1 = 0 \\ A + B = 0 \end{array}\right| \begin{array}{l} A = 1 \\ B = -1. \end{array}$$

Thus

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

and we could proceed with the partial sums ...

This is a special case of the partial fraction decomposition.

Let

$$f(x) = \frac{P(x)}{\varphi(x)}, \qquad \deg P < \deg \varphi.$$

- The denom. $\varphi(x)$ can always be factorised as a product of polynomials of 1^{st} and 2^{nd} degree, say

$$\varphi(x) = (a_1 x + b_1)^{n_1} \cdot \ldots \cdot (a_k x + b_k)^{n_k} \cdot$$
$$\cdot (c_1 x^2 + d_1 x + e_1)^{m_1} \cdot \ldots \cdot (c_\ell x^2 + d_\ell x + e_\ell)^{m_\ell}.$$

- We want to decompose $\frac{P(x)}{\varphi(x)}$ as a sum of fractions with denominators

$$a_1 x + b_1, \ldots, a_k x + b_k, c_1 x^2 + d_1 x + e_1, \ldots, c_\ell x^2 + d_\ell x + e_\ell$$

- For each factor $(ax+b)^n$ we want the terms

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$$

to appear in the sum, while for each factor $(cx^2+dx+e)^m$ we want the terms

$$\frac{A_1 x + B_1}{cx^2+dx+e} + \frac{A_2 x + B_2}{(cx^2+dx+e)^2} + \ldots + \frac{A_m x + B_m}{(cx^2+dx+e)^m}$$

to appear (here $A_1, \ldots, A_m, B_1, \ldots, B_m$ are coefficients to be found).

E.g. Decompose $\dfrac{1}{(x+1)(x^2+x+1)}$ into partial fractions.

We want to find $A, B, C \in \mathbb{R}$ such that

$$\frac{1}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} \quad \forall x \neq -1 \Rightarrow$$

$$A(x^2+x+1) + (Bx+C)(x+1) = 1 \quad \forall x \neq -1 \Rightarrow$$

$$Ax^2 + Ax + A + Bx^2 + Bx + Cx + C = 1 \quad \forall x \neq -1 \Rightarrow$$

$$(A+B)x^2 + (A+B+C)x + (A+C-1) = 0 \quad \forall x \neq -1$$

$$\left.\begin{array}{l} A + B = 0 \\ A + B + C = 0 \\ A + C - 1 = 0 \end{array}\right| \begin{array}{l} A = 1 \\ B = -1 \\ C = 0 \end{array}$$

Therefore

$$\frac{1}{(x+1)(x^2+x+1)} = \frac{1}{x+1} - \frac{x}{x^2+x+1}.$$

E.g. Examine the convergence of $\sum_{n=2}^{\infty} \frac{1}{n(n-1)(n+2)}$.

We want to find $A, B, C \in \mathbb{R}$ such that

$$\frac{1}{n(n-1)(n+2)} = \frac{A}{n} + \frac{B}{n-1} + \frac{C}{n+2} \quad \forall n \geq 2 \Rightarrow$$

$$A(n-1)(n+2) + Bn(n+2) + Cn(n-1) - 1 = 0 \quad \forall n \geq 2 \Rightarrow$$

$$A(x-1)(x+2) + Bx(x+2) + Cx(x-1) = 1 \quad \forall x \in \mathbb{R}$$

For $x=0$: $\quad A \cdot (-1) \cdot 2 = 1 \Rightarrow A = -1/2$

For $x=1$: $\quad 3B = 1 \Rightarrow B = 1/3$

For $x=-2$: $\quad (-2) \cdot (-3) C = 1 \Rightarrow C = 1/6$.

Thus

$$\frac{1}{n(n-1)(n+2)} = \frac{1}{3} \cdot \frac{1}{n-1} + \frac{1}{6} \cdot \frac{1}{n+2} - \frac{1}{2} \cdot \frac{1}{n}$$

$$= \frac{1}{3} \cdot \left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{1}{6}\left(\frac{1}{n+2} - \frac{1}{n}\right)$$

$$= \frac{1}{3} \cdot \left(\frac{1}{n-1} - \frac{1}{n}\right) - \frac{1}{6}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

The N-th partial sum of the series is

$$S_N = \sum_{n=2}^{N} \frac{1}{n(n-1)(n+2)} = \frac{1}{3} \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n}\right) - \frac{1}{6} \sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \frac{1}{3}\left(1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\cdots} + \cancel{\frac{1}{N-1}} - \frac{1}{N}\right)$$

$$- \frac{1}{6}\left(\frac{1}{2} - \cancel{\frac{1}{4}} + \frac{1}{3} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \frac{1}{6} + \cancel{\cdots} + \cancel{\frac{1}{N}} - \frac{1}{N+2}\right)$$

$$= \frac{1}{3}\left(1 - \frac{1}{N}\right) - \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2}\right),$$

so when $N \to \infty$,

$$\lim_{N\to\infty} S_N = \frac{1}{3} \cdot 1 - \frac{1}{6} \cdot \left(\frac{1}{2} + \frac{1}{3}\right) = \frac{7}{36}.$$

Thus $$\sum_{n=2}^{\infty} \frac{1}{n(n-1)(n+2)} = \frac{7}{36}.$$

So far we have seen series of the form $\sum_{n=1}^{\infty} a_n$, where $(a_n)_{n=1}^{\infty}$ is a fixed sequence of real numbers.

A series of the form $\sum_{n=0}^{\infty} a_n x^n$ $(x \in \mathbb{R})$ is called a <u>power series.</u>

Power series are functions of the variable x:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad (*)$$

A power series might converge for some values of $x \in \mathbb{R}$ and diverge for some others. E.g. the power series in $(*)$ converges for $x=0$.

*A power series $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$

can be viewed as "a polynomial of infinite degree".

There exist functions with very well-known power series expansions:

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad x \in \mathbb{R}.$

2. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \quad x \in \mathbb{R}$

3. $\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \quad x \in \mathbb{R}$

4. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots, \quad -1 < x < 1.$

Consider, for example, the first expansion:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}.$$

This means that:

- the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$
- for all $x \in \mathbb{R}$, it converges to e^x.

The fourth power series $\sum_{n=0}^{\infty} x^n$

converges to $\frac{1}{1-x}$ for all $x \in (-1,1)$.

That means, that if I consider the function

$$f(x) = \frac{1}{1-x}, \quad x \in (-\infty, 1) \cup (1, \infty)$$

then $f(x)$ has a power series expansion for all $x \in (-1,1)$, so not for all x in its domain of definition.

In other words, the function

$$g(x) = \sum_{n=0}^{\infty} x^n$$

is well-defined for all $-1 < x < 1$ and $g(x) = f(x)$ for all $-1 < x < 1$ (but not for all x in the domain of f).

The power series of the form $\sum_{n=0}^{\infty} a_n x^n$ is called a <u>Taylor series</u> around 0 (or also a McLaurin series).

If $f(x)$ has a Taylor series $\sum_{n=0}^{\infty} a_n x^n$ then its Taylor polynomial of degree N will be $\sum_{n=0}^{N} a_n x^n$.

ASIDE.