REMARK: The same result is true for the improper integrals

$$\int_a^{+\infty} \frac{dx}{x^p} \quad \text{for any } a > 0.$$

They converge if and only if $p > 1$ – but the value to which they converge depends on a.

Exercise: Study the behaviour (convergence / divergence) of the improper integral

$$\int_0^1 \frac{dx}{x^q} \quad \text{for the different values of } q > 0.$$

- When $q > 1$,

$$\int_u^1 \frac{dx}{x^q} = \int_u^1 x^{-q}\,dx = \left[\frac{x^{1-q}}{1-q}\right]_u^1$$

$$= \left[-\frac{1}{(q-1)\,x^{q-1}}\right]_u^1$$

$$= \frac{1}{(q-1)\,u^{q-1}} - \frac{1}{q-1} \xrightarrow{u \to 0^+} +\infty$$

and the integral diverges.

when $q=1$,

$$\int_u^1 \frac{dx}{x} = [\ln x]_u^1 = -\ln u \xrightarrow{u \to 0^+} +\infty$$

and the integral diverges.

When $0<q<1$,

$$\int_u^1 x^{-q}dx = \left[\frac{x^{1-q}}{1-q}\right]_u^1 = \frac{1}{1-q} - \frac{u^{1-q}}{1-q} \xrightarrow{u \to 0^+} \frac{1}{1-q}$$

and $\int_0^1 \frac{dx}{x^q}$ converges to $\frac{1}{1-q}$.

The improper integral

$\int_0^1 \frac{1}{x^q}dx$ converges if and only if $q<1$.

* We did not examine the case $q<0$ at all, because convergence is then trivial — the integral

$$\int_0^1 x^{-q}dx$$

is a Riemann integral and not an improper integral.

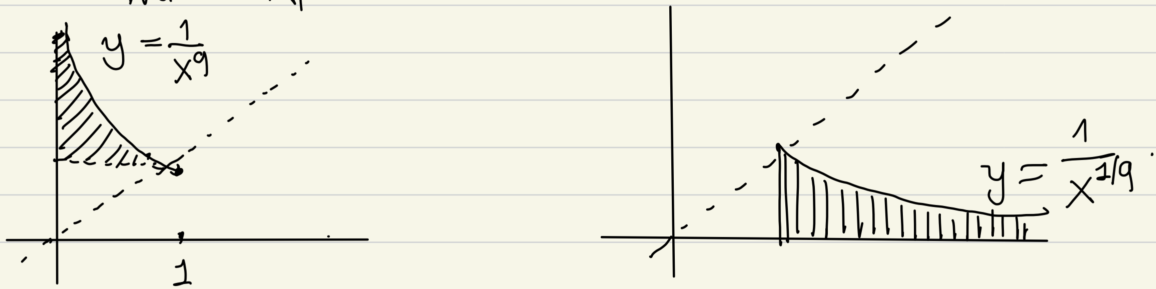
REMARK: Observe that

$$\int_1^\infty \frac{dx}{x^p} < \infty \iff p > 1$$

while, on the other hand,

$$\int_0^1 \frac{dx}{x^q} < \infty \iff q < 1.$$

This was expected.



The inverse of the function $f(x) = \frac{1}{x^q}$ is the function $g(x) = x^{-\frac{1}{q}}$. The area on the figure on the left is "equal" to the area on the right.

The improper integral of $1/x^{1/q}$ converges if and only $\frac{1}{q} > 1 \iff q < 1$.

- For which values of $\lambda > 0$ does $\int_0^{\infty} e^{-\lambda x}\,dx$ converge?

ANSWER: For $\lambda > 0$,

$$\int_0^T e^{-\lambda x}\,dx = \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^T$$

$$= \frac{1}{\lambda} - \frac{1}{\lambda e^{\lambda T}} \xrightarrow{T \to +\infty} \frac{1}{\lambda}$$

so $\int_0^{\infty} e^{-\lambda x}\,dx = \frac{1}{\lambda}$ for any $\lambda > 0$.

<u>Comment</u>: This should also be expected. For any $\lambda > 0$,

$$\lim_{x \to +\infty} \frac{e^{-\lambda x}}{\frac{1}{x^2}} = \lim_{x \to +\infty} \frac{x^2}{e^{\lambda x}} = 0$$

so heuristically $e^{-\lambda x}$ is much "smaller" than $\frac{1}{x^2}$. And since $\int_1^{\infty} \frac{1}{x^2}\,dx < \infty$, we expect that so does $\int_0^{\infty} e^{-\lambda x}\,dx$.

THEOREM 5.17: Suppose $f, g : [a, \infty) \to \mathbb{R}$ are such that

$$0 \leq f(x) \leq g(x) \quad \text{for all } x \geq x_0$$

(for some $x_0 \geq a$), then:

(i) if $\int_a^{+\infty} g(x)\,dx$ converges, then so does $\int_a^{+\infty} f(x)\,dx$.

(ii) if $\int_a^{+\infty} f(x)\,dx$ diverges, then so does $\int_a^{+\infty} g(x)\,dx$.

(The same is true for type II - improper integrals)

This theorem helps us decide if an improper integral converges or not, even if we cannot find its value.

Apparently in order to use Theorem 5.17 we need to know basic improper integrals which converge or diverge.

- $\int_1^{\infty} \frac{1}{x^p}\,dx < \infty$ when $p > 1$
- $\int_1^{\infty} e^{-\lambda x}\,dx < \infty$ for any $\lambda > 0$

- $\displaystyle\int_0^1 \frac{dx}{x^q} < \infty$ when $q < 1$

- $\displaystyle\int_0^1 \ln x \, dx < \infty$

Examples

(i) $\displaystyle\int_0^{+\infty} \frac{1+x^2}{1+x^4}\, dx$?

For $x > 1$,

$$\frac{x^2+1}{x^4+1} = \frac{x^2\cdot\left(1+\frac{1}{x^2}\right)}{x^4+1} < \frac{x^2\left(1+\frac{1}{1}\right)}{x^4} = \frac{2}{x^2}$$

and $\displaystyle\int_1^{\infty} \frac{2}{x^2}\, dx < \infty$, hence so does

$$\int_1^{\infty} \frac{1+x^2}{1+x^4}\, dx$$

and also

$$\int_0^{\infty} \frac{1+x^2}{1+x^4}\, dx.$$

(ii) $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}+1}{x^2+1}$?

For $x > 1$,

$$\sqrt{x}+1 < \sqrt{x}+\sqrt{x} = 2\sqrt{x} \quad \text{and}$$

$$x^2+1 > x^2 \Rightarrow \frac{1}{x^2+1} < \frac{1}{x^2}$$

hence

$$\frac{\sqrt{x}+1}{x^2+1} < \frac{2}{x^{3/2}}.$$

Since $\displaystyle\int_1^{\infty} \frac{dx}{x^{3/2}}$ converges,

so does $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}+1}{x^2+1}\,dx$.

* Suppose $\int_a^b f(x)dx$ is improper at both a and b.

We examine

$$I_1 = \int_a^{x_0} f(x)dx \quad \text{and} \quad \int_{x_0}^b f(x)dx = I_2.$$

If the int. converges, the values of I_1 and I_2 depend on x_0 but the sum

$$I_1 + I_2$$

will not depend on x_0.

(iii) $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x+x^3}}$?

We examine $\displaystyle\int_0^1 \frac{dx}{\sqrt{x+x^3}}$ and $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x+x^3}}$.

For $0 < x \leq 1$,

$$\sqrt{x+x^3} = \sqrt{x} \cdot \sqrt{1+x^2} > \sqrt{x} \Longrightarrow$$

$$\frac{1}{\sqrt{x+x^3}} < \frac{1}{\sqrt{x}}$$

and $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} < \infty$,

hence so does $\displaystyle\int_0^1 \frac{dx}{\sqrt{x+x^3}}$.

For $x \geq 1$,

$$x + x^3 > x^3 \Rightarrow$$

$$\frac{1}{\sqrt{x+x^3}} < \frac{1}{x^{3/2}}$$

and

$$\int_1^{\infty} \frac{1}{x^{3/2}}\, dx < \infty$$

hence so does $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x+x^3}}\, dx$.

Therefore $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x+x^3}} < \infty$.

(iv) $$\int_0^\infty \frac{dx}{\sqrt{x+x^2}}$$

For $x > 1$,

$$x + x^2 < 2x^2 \Rightarrow$$

$$\frac{1}{\sqrt{x+x^2}} > \frac{1}{\sqrt{2}\,x}$$

and $\int_1^\infty \frac{1}{\sqrt{2}\,x}\,dx$ diverges,

hence so does

$$\int_1^\infty \frac{1}{\sqrt{x+x^2}}\,dx.$$

Therefore $$\int_0^\infty \frac{dx}{\sqrt{x+x^2}} = \infty.$$

Recall that in order to apply Theorem 5.17 for an integral

$$\int_a^\infty f(x)\,dx$$

we only need $0 \leq f(x) \leq g(x)$, $x > x_0$.
But if, in addition, we have

$$0 \leq f(x) \leq g(x) \quad \text{for all } x \geq a$$

then we may deduce

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$$

Example: Prove that $\int_0^\infty e^{-x^2}dx$ converges, and also

$$\int_0^\infty e^{-x^2}dx < 1 + \frac{1}{e}.$$

- For $x > 1$,

$$x^2 > x \Rightarrow -x^2 < -x \Rightarrow e^{-x^2} < e^{-x}$$

and $\int_1^\infty e^{-x}dx < \infty$, hence so does $\int_1^\infty e^{-x^2}dx$

Thus $\int_0^\infty e^{-x^2}dx$ converges.

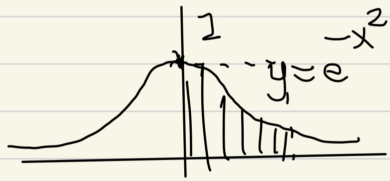
$$\int_0^{\infty} e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^{+\infty} e^{-x^2}\,dx$$

For $0 \leqslant x \leqslant 1$, we have

$$-x^2 < 0 \Rightarrow e^{-x^2} < 1$$

$$\Rightarrow \int_0^1 e^{-x^2}\,dx < 1.$$

For $x \geqslant 1$,



$$e^{-x^2} < e^{-x} \Rightarrow$$

$$\int_1^{\infty} e^{-x^2}\,dx < \int_1^{\infty} e^{-x}\,dx$$

$$= \lim_{T \to +\infty} \int_1^T e^{-x}\,dx$$

$$= \lim_{T \to +\infty} \left(\frac{1}{e} - \frac{1}{e^T}\right)$$

$$= \frac{1}{e}.$$

Hence $\int_0^{\infty} e^{-x^2}\,dx < 1 + \frac{1}{e}$.

REMARK: Theorem 5.17 only covers the case when $f(x), g(x)$ are both positive.

When both $f(x), g(x) < 0$
Theorem 5.17 implies that whenever

$$g(x) \leq f(x) < 0 \quad \text{for all } x$$

then

(i) if $\int_a^b g(x)\,dx$ converges,

then so does $\int_a^b f(x)\,dx$.

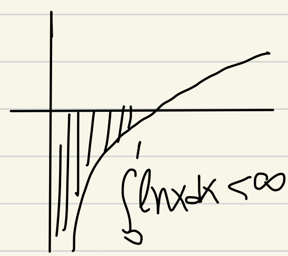
(ii) if $\int_a^b f(x)\,dx$ diverges,

then so does $\int_a^b g(x)\,dx$.

E.g. Does $\int_0^{\pi/2} (\sin x)\cdot(\ln x)\,dx$ converge?

For $0 < x < 1$,

$$0 < \sin x < 1 \overset{\ln x < 0}{\Longrightarrow}$$
$$0 > \sin x \cdot \ln x > \ln x \Longrightarrow$$
$$\ln x < \sin x \cdot \ln x < 0$$



and $\int_0^1 \ln x\,dx$ converges

hence so does $\int_0^1 \sin x \cdot \ln x\,dx$

and also $\int_0^{\pi/2} \sin x \cdot \ln x\,dx$.