THEOREM 5.15 (Mean Value Theorem of Integral Calculus): If $f:[a,b]\to\mathbb{R}$ is continuous, there exists $\xi\in(a,b)$ such that

$$f(\xi)=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

PROOF

Define

$$F(x)=\int_a^x f(t)\,dt, \qquad x\in[a,b].$$

F is continuous on $[a,b]$ and differentiable on (a,b), so by the Mean Value Theorem there exists $\xi\in(a,b)$ such that

$$F'(\xi)=\frac{F(b)-F(a)}{b-a} \Rightarrow$$

$$f(\xi)=\frac{1}{b-a}\int_a^b f(x)\,dx. \qquad \blacksquare$$

REMARK: In the literature, Theorem 5.15 is proved using only the definition of Riemann integration, and it is then used for the proof of the Fundamental Theorem of Integral Calculus.

Here we followed the opposite direction.

IMPROPER INTEGRALS

We have defined the Riemann integral of functions $f:[a,b]\to\mathbb{R}$ on intervals $[a,b]$ where $a,b\in\mathbb{R}$.

We want to define integrals of functions on intervals $(a,b]$, $[a,b)$, (a,b) when:

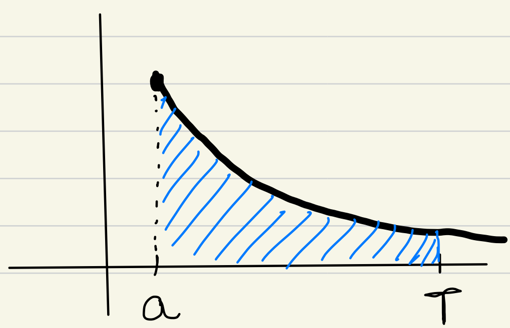
- at least one of a,b is $\pm\infty$, or
- f cannot be extended to a Riemann integrable function $F:[a,b]\to\mathbb{R}$.

Improper Integrals of Type I

- Let $f:[a,\infty)\to\mathbb{R}$ be Riemann-int. on every interval of the form $[a,T]$, $T>a$.

We say that the improper integral $\int_a^{+\infty} f(x)\,dx$ converges to the real number I if $\lim_{T\to\infty}\int_a^T f(x)\,dx = I$.

Otherwise we say that $\int_a^{+\infty} f(x)\,dx$ diverges.



In the special case when
$$\lim_{T\to+\infty} \int_a^T f(x)\,dx = \pm\infty$$
we say that $\int_a^{+\infty} f(x)\,dx$ diverges to $\pm\infty$.

When $\int_a^{+\infty} f(x)\,dx$ converges to $I \in \mathbb{R}$

we may write $\int_a^{+\infty} f(x)\,dx = I$.

* When we say that f cannot be extended to a Riemann-int. $F:[a,b]\to\mathbb{R}$ we mean that there is no function
$$F:[a,b]\to\mathbb{R}$$
which is Riemann-integr. and
$$F(x) = f(x) \quad \text{for all} \quad x\in(a,b).$$

- A similar definition holds for improper integrals of the form
$$\int_{-\infty}^{a} f(x)\,dx.$$

- We say that the improper integral $\int_{-\infty}^{+\infty} f(x)\,dx$ converges to $I\in\mathbb{R}$,

if there exists some $a\in\mathbb{R}$ such that

- We say that the improper integral $\int_{-\infty}^{+\infty} f(x)\,dx$ converges to $I \in \mathbb{R}$, if there exists some $a \in \mathbb{R}$ such that both integrals

$$\int_a^{+\infty} f(x)\,dx = I_1 \quad , \quad \int_{-\infty}^{a} f(x)\,dx = I_2$$

and $I_1 + I_2 = I$.

Examples

(i) $\displaystyle\int_0^{+\infty} \frac{\arctan x}{1+x^2}\,dx$.

$$\int_0^T \frac{\arctan x}{1+x^2}\,dx = \int_0^T \arctan x \cdot (\arctan x)'\,dx$$

$$= \left[\frac{1}{2}\arctan^2 x\right]_0^T$$

$$= \frac{1}{2}\arctan^2 T \xrightarrow{T\to\infty} \frac{1}{2}\cdot\frac{\pi^2}{4} = \frac{\pi^2}{8}.$$

Therefore $\displaystyle\int_0^{\infty} \frac{\arctan x}{1+x^2}\,dx = \frac{\pi^2}{8}$.

(ii) $\displaystyle\int_1^{+\infty} \frac{dx}{x}$.

$$\int_1^T \frac{dx}{x} = \ln T \xrightarrow{T\to\infty} +\infty$$

The integral $\displaystyle\int_1^{+\infty} \frac{dx}{x}$ diverges to $+\infty$.

(iii) $\displaystyle\int_0^{+\infty} \cos x \, dx$

$$\int_0^T \cos x \, dx = [\sin x]_0^T = \sin T$$

The limit $\displaystyle\lim_{T\to\infty} \int_0^T \cos x \, dx$ does not exist,

therefore $\displaystyle\int_0^{\infty} \cos x \, dx$ diverges

(but not to ∞ !!)

(iv) $\displaystyle\int_{-\infty}^0 e^x \, dx$

$$\int_T^0 e^x \, dx = [e^x]_T^0 = 1 - e^T \xrightarrow{T\to-\infty} 1$$

Thus $\displaystyle\int_{-\infty}^0 e^x \, dx = 1$.

(v) $\int_{-\infty}^{+\infty} x\,dx$

We have to examine $\int_0^{\infty} x\,dx$, $\int_{-\infty}^{0} x\,dx$.

$$\int_0^T x\,dx = \frac{T^2}{2} \to +\infty$$

One of the two impr. integrals diverges, therefore

$$\int_{-\infty}^{+\infty} x\,dx \quad \text{also diverges.}$$

* Observe that $\int_{-\infty}^{+\infty} x\,dx$ diverges even though

$$\lim_{T\to+\infty} \int_{-T}^{T} x\,dx = 0.$$

Therefore the existence of

$$\lim_{T\to+\infty} \int_{-T}^{T} f(x)\,dx$$

does not imply convergence of $\int_{-\infty}^{+\infty} f(x)\,dx$.

(vii) $\displaystyle\int_{-\infty}^{+\infty} \frac{\arctan x}{1+x^2}\,dx$

We need to examine

$$\int_0^{+\infty} \frac{\arctan x}{1+x^2}\,dx \quad , \quad \int_{-\infty}^{0} \frac{\arctan x}{1+x^2}\,dx .$$

We saw that $\displaystyle\int_0^{\infty} \frac{\arctan x}{1+x^2}\,dx = \frac{\pi^2}{8}$.

Similarly we can show that

$$\int_{-\infty}^{0} \frac{\arctan x}{1+x^2}\,dx = -\frac{\pi^2}{8} .$$

Therefore $\displaystyle\int_{-\infty}^{+\infty} \frac{\arctan x}{1+x^2}\,dx$ converges to 0.

* If we choose to study the integrals

$\displaystyle\int_a^{+\infty}$ and $\displaystyle\int_{-\infty}^{a}$, $a \neq 0$

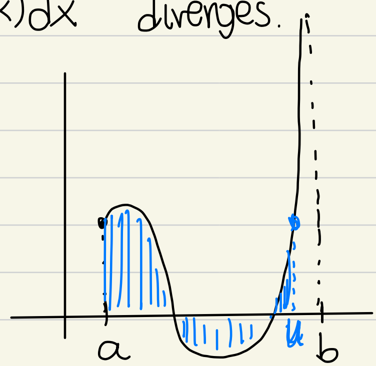
their values (of these 2 integrals) would be different, but their sum would stile be 0 – so the result we find does not depend on the choice of a.

Improper Integrals of Type II

- Let $f : [a, b) \to \mathbb{R}$ be Riemann-integrable on every interval of the form $[a, u]$, $a < u < b$ and f cannot be extended to a Riemann-integrable function $F : [a, b] \to \mathbb{R}$.

We say that the improper integral $\int_a^b f(x)dx$ converges to the real number I if
$$\lim_{u \to b^-} \int_a^u f(x)\,dx = I.$$

Otherwise we say that $\int_a^b f(x)dx$ diverges.



In the special case when
$$\lim_{u \to b^-} \int_a^u f(x)dx = \pm\infty$$
we say that $\int_a^b f(x)dx$ diverges to $\pm\infty$.

- A similar definition when $f : (a, b] \to \mathbb{R}$ cannot be extended to a R.-int. $F : [a, b] \to \mathbb{R}$.

- If $f:(a,b)\to\mathbb{R}$ is Riemann-int. on any interval $[A,B]$, with $a<A<B<b$ and f cannot be extended to any Riemann integrable F on any interval of the form $[a,A]$, $[B,b]$,

we say that the improper int. $\int_a^b f(x)\,dx$ converges to $I\in\mathbb{R}$ if there exists some $x_0\in(a,b)$ with

$$\int_{x_0}^b f(x)\,dx = I_1 \,, \quad \int_a^{x_0} f(x)\,dx = I_2 \quad \text{and} \quad I_1+I_2=I.$$

Otherwise we say that $\int_a^b f(x)\,dx$ diverges.

<u>Examples</u>

(i) $\displaystyle\int_0^1 \frac{dx}{1-x}$

$$\int_0^u \frac{dx}{1-x} = \Big[-\ln|1-x|\Big]_0^u$$

$$= \ln\frac{1}{|1-u|} \xrightarrow{u\to 1^-} +\infty$$

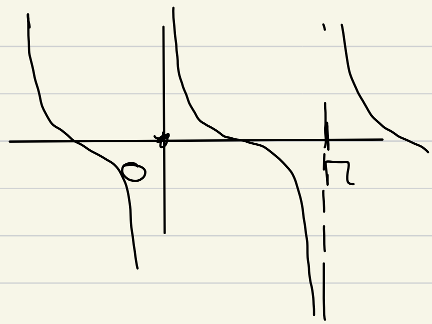
Hence $\displaystyle\int_0^1 \frac{dx}{1-x}$ diverges to $+\infty$.

(ii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

$$\int_y^1 \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_y^1 = 2 - 2\sqrt{y} \xrightarrow{y \to 0^+} 2$$

Hence $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = 2.$

(iii) $\displaystyle\int_0^{\pi} \cot x \, dx$



We examine

$\displaystyle\int_0^{\pi/2} \cot x \, dx$, $\displaystyle\int_{\pi/2}^{\pi} \cot x \, dx$.

$$\int_u^{\pi/2} \cot x \, dx = \int_u^{\pi/2} \frac{\cos x}{\sin x} \, dx$$

$$= \left[\ln|\sin x|\right]_u^{\pi/2}$$

$$= \ln\left|\sin\frac{\pi}{2}\right| - \ln|\sin u|$$

$$= -\ln|\sin u| \xrightarrow{u \to 0^+} +\infty$$

Therefore $\displaystyle\int_0^{\pi} \cot x \, dx$ diverges.

(iv) $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$

We examine $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$, $\int_{-1}^{0} \frac{dx}{\sqrt{1-x^2}}$.

$$\int_{0}^{u} \frac{dx}{\sqrt{1-x^2}} = \left[\arcsin x\right]_0^u$$
$$= \arcsin u - \cancel{\arcsin 0}$$
$$\xrightarrow{u \to 1^-} \frac{\pi}{2}$$

$$\int_{u}^{0} \frac{dx}{\sqrt{1-x^2}} = -\arcsin u \xrightarrow{u \to -1^+} \frac{\pi}{2}$$

Hence $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$.

* Sometimes we might encounter improper integrals of "mixed-type". We treat them as in the third case of Type I and Type II improper integrals.

E.g. $\int_{0}^{+\infty} \frac{\ln x}{x} dx$.

We take $\int_{e}^{+\infty} \frac{\ln x}{x} dx$, $\int_{0}^{e} \frac{\ln x}{x} dx$.

(it diverges).

PROPOSITION 5.16: The improper integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges when $p>1$ and diverges when $0<p\leq 1$.

PROOF

When $p>1$,

$$\int_1^T \frac{dx}{x^p} = \int_1^T x^{-p}\, dx = \left[\frac{x^{1-p}}{1-p}\right]_1^T$$

$$= \left[-\frac{1}{p-1}\cdot\frac{1}{x^{p-1}}\right]_1^T$$

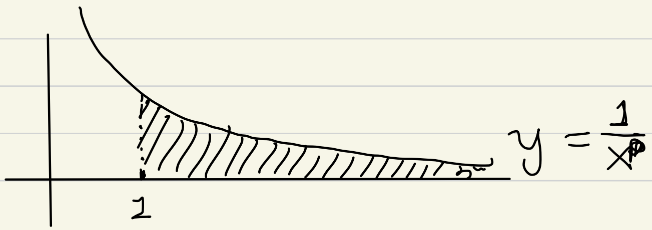
$$= \frac{1}{p-1} - \frac{1}{p-1}\cdot\frac{1}{T^{p-1}} \longrightarrow \frac{1}{p-1}$$

When $p=1$,

$$\int_1^T \frac{dx}{x} = \ln T \xrightarrow{T\to\infty} +\infty$$

When $0<p<1$

$$\int_1^T \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^T = \frac{T^{1-p}}{1-p} - \frac{1}{1-p} \xrightarrow{T\to\infty} +\infty$$



REMARK: The same result is true for the improper integrals

$$\int_a^{+\infty} \frac{dx}{x^p} \quad \text{for any } a>0.$$

They converge if and only if $p>1$ – but the value to which they converge depends on a.

Exercise: Study the behaviour (convergence / divergence) of the improper integral

$$\int_0^1 \frac{dx}{x^q} \quad \text{for the different values of } q>0.$$