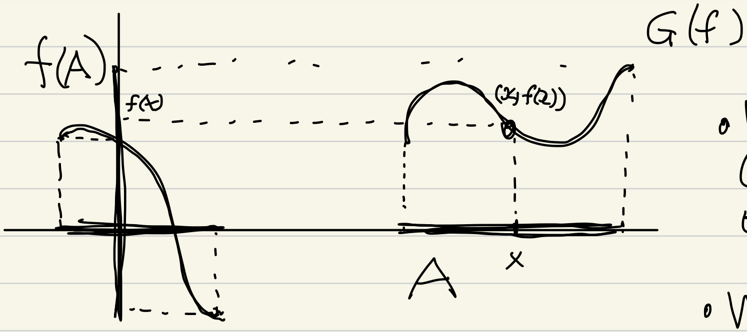
Here we deal with real functions of a real variable, that is, functions of the form
$$f : A \to \mathbb{R}, \quad \text{where } A \subseteq \mathbb{R}.$$

- When $f : A \to \mathbb{R}$ is a function, then f stands for the function itself, but $f(x)$ is a real number. (These are 2 different things)
  However, very often we say "the function $f(x)$" to refer to the function
  $$f : A \to \mathbb{R},$$
  $$x \mapsto f(x).$$

- Suppose we are given some function f. We assume that the domain of f is the largest possible subset of $\mathbb{R}$ such that $f(x)$ is well-defined (unless otherwise stated).
  E.g. when we are given $f(x) = \sqrt{x-1}$ then we assume that $D_f = [1, \infty)$.
  Of course, someone could define the function
  $$g : [2,3] \to \mathbb{R},$$
  $$g(x) = \sqrt{x-1}, \quad x \in [2,3].$$

If $f: A \to \mathbb{R}$ is a function.
The graph of f is the set

$$G(f) = \{ (x, f(x)) : x \in A \} \subseteq \mathbb{R}^2 .$$



- When we project $G(f)$ on the horizontal axis, we get A.
- When we project $G(f)$ on the vertical axis, we get $f(A)$

Given two functions $f: D_f \to \mathbb{R}$, $g: D_g \to \mathbb{R}$
we can define new functions

$$f+g, \quad f-g, \quad f \cdot g, \quad \frac{f}{g}$$

as follows:

$$f+g : D_f \cap D_g \to \mathbb{R}, \quad (f+g)(x) = f(x) + g(x)$$

$$f-g : D_f \cap D_g \to \mathbb{R}, \quad (f-g)(x) = f(x) - g(x)$$

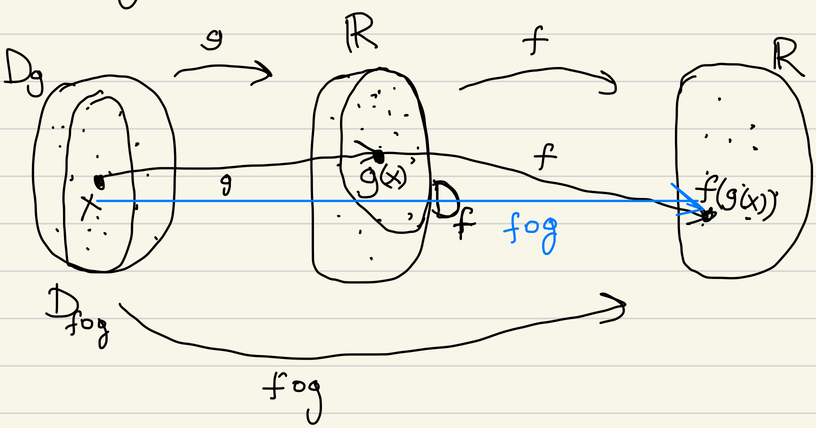
$$f \cdot g : D_f \cap D_g \to \mathbb{R}, \quad (f \cdot g)(x) = f(x) \cdot g(x)$$

$$\frac{f}{g} : (D_f \cap D_g) \setminus \{x : g(x) = 0\} \to \mathbb{R}, \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} .$$

Let $f: D_f \to \mathbb{R}$, $g: D_g \to \mathbb{R}$ be two functions. We define the composition of f with g and we denote it as $f \circ g$, as follows:

$$D_{f\circ g} = \{ x \in D_g : g(x) \in D_f \},$$

$$(f\circ g)(x) = f(g(x)) \quad \text{for all } x \in D_{f\circ g}.$$



E.g. if $f(x) = \frac{1}{x}$, $g(x) = \ln x$

then we can find $f\circ g$ and $g\circ f$.

$D_f = \mathbb{R}\setminus\{0\} = (-\infty, 0) \cup (0, \infty)$.

$D_g = (0, +\infty)$.

- $D_{f\circ g} = \{ x \in D_g : g(x) \in D_f \}$

$= \{ x \in (0,\infty) : \ln x \neq 0 \}$

$= \{ x \in (0,\infty) : x \neq 1 \}$

$= (0,1) \cup (1, +\infty)$.

$$(f \circ g)(x) = f(g(x)) = \frac{1}{\ln x} \quad \text{for all } x \in (0,1) \cup (1,\infty).$$

- $$D_{g \circ f} = \{ x \in D_f : f(x) \in D_g \}$$
$$= \{ x \neq 0 : \frac{1}{x} > 0 \}$$
$$= (0, \infty).$$

$$(g \circ f)(x) = g(f(x)) = \ln \frac{1}{x} = -\ln x, \quad \text{for all } x > 0.$$

This shows that $f \circ g$ and $g \circ f$ are not in general the same function. (In the previous example, they did not even have the same domain of definition).

A function $f : X \to Y$ is called <u>1-1</u> (or <u>invertible</u>) if for all $x_1, x_2 \in X$:

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

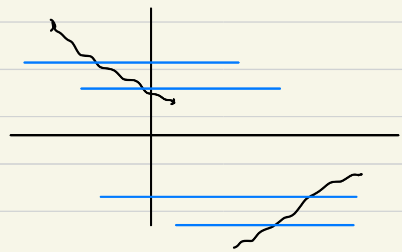
(I.e. different elements in the domain X are mapped onto different elements in $f(X)$).

E.g. take $h : \mathbb{R} \to \mathbb{R}$, $h(x) = x^4$

This is not 1-1, because

$$2 \neq -2 \quad \text{but} \quad h(2) = h(-2) = 16.$$

When we have the graph of a real function $f: A \to \mathbb{R}$, we can understand if it is 1-1 or not as follows:



f is 1-1 if and only if every line parallel to the x-axis intersects the graph of f at at most one point.

THIS FUNCTION IS NOT 1-1.

$a \neq b$ BUT $f(a) = f(b)$

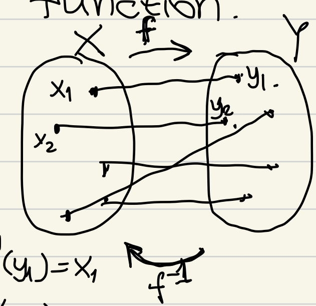
* A function f is not 1-1 if there exist $x_1, x_2 \in D_f$ with

$$x_1 \neq x_2 \text{ and } f(x_1) = f(x_2).$$

Suppose $f: X \to Y$ is a 1-1 function. Then we can define a new function

$$f^{-1}: f(X) \to X$$

which maps every $y \in f(X)$ to the unique $x \in X$ such that $f(x) = y$.



$y_1 = f(x_1)$

$y_2 = f(x_2)$

$\vdots$

$f^{-1}(y_1) = x_1$

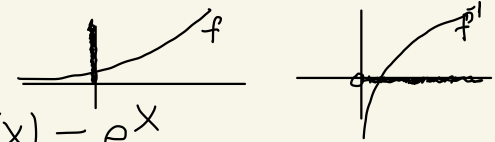
$f^{-1}(y_2) = x_2$

$\vdots$

$$f^{-1}: f(X) \to X$$

$$f^{-1}(y) = x \iff f(x) = y.$$

The function f^{-1} is called the <u>inverse</u> function of f.



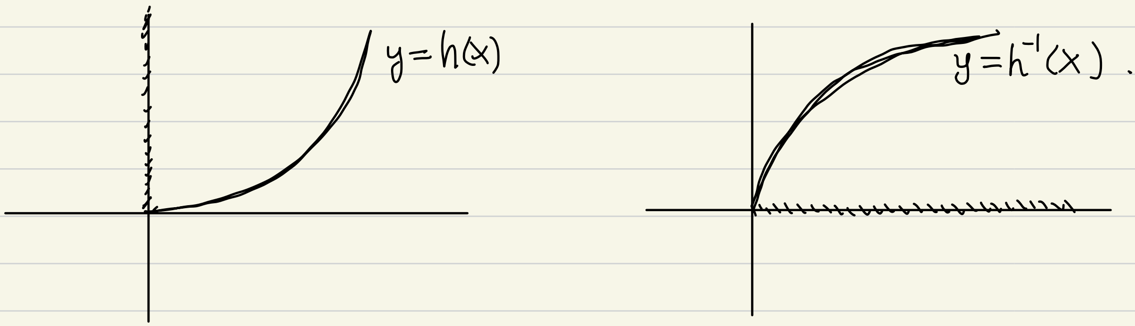
E.g. if $f : \mathbb{R} \to (0,\infty)$, $f(x) = e^x$
Then $f^{-1} : (0,\infty) \to \mathbb{R}$, $f^{-1}(x) = \ln x$.

Also if $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$, $x \in \mathbb{R}$.
then g is NOT 1-1 $(g(1) = g(-1))$.
But if we consider

$$h : [0,\infty) \to [0,\infty), \quad h(x) = x^2$$

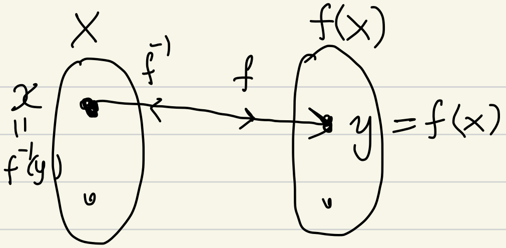
then this function is 1-1, and its inverse function is

$$h^{-1} : [0,\infty) \to [0,\infty), \quad h^{-1}(x) = \sqrt{x}.$$



Generally, if $f : X \to Y$ is 1-1
(so that $f^{-1} : f(X) \to X$ is well-defined)
then

$$f^{-1}(f(x)) = x \qquad \text{for all } x \in X$$
$$f(f^{-1}(y)) = y \qquad \text{for all } y \in f(X).$$



$$y = f(x) \iff x = f^{-1}(y)$$

$$f^{-1}(f(x)) = x \quad \text{for all } x \in X$$
$$f(f^{-1}(y)) = y \quad \text{for all } y \in f(X).$$

The two last relations show that the compositions $f^{-1} \circ f$ and $f \circ f^{-1}$ are equal to the "identity function" (the function which maps every element on itself) on the sets X and $f(X)$, respectively.

So f^{-1} is the "inverse" of f with respect to the operation "$\circ$" of composition of functions.

<u>Example</u>: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x+1} - 1$.

f is invertible because

$$f(x_1) = f(x_2) \iff e^{2x_1+1} - 1 = e^{2x_2+1} - 1$$
$$\iff e^{2x_1+1} = e^{2x_2+1}$$
$$\iff x_1 = x_2.$$

What is the range of f?

$$y = f(x) \iff y = e^{2x+1} - 1$$
$$\iff \boxed{y + 1 = e^{2x+1}}$$

This can have a solution in y if and only if $y + 1 > 0 \iff y > -1$.

This means that $f(\mathbb{R}) = (-1, +\infty)$.

For any $y > -1$,

$$y = f(x) \iff y + 1 = e^{2x+1}$$

$$\iff 2x + 1 = \ln(y+1)$$

$$\iff x = \frac{1}{2}\ln(y+1) - \frac{1}{2}.$$

So the inverse of f is

$$f^{-1} : (-1, \infty) \to \mathbb{R}, \quad f^{-1}(y) = \frac{1}{2}\ln(y+1) - \frac{1}{2}.$$

2. LIMITS AND CONTINUITY

• UPPER & LOWER BOUNDS - SUPREMUM & INFIMUM

Let $A \subseteq \mathbb{R}$. We say that A is

(i) <u>bounded from above</u> if there exists some $M \in \mathbb{R}$ such that
$$a \leq M \quad \text{for any } a \in A.$$

(ii) <u>bounded from below</u> if there exists some $m \in \mathbb{R}$ such that
$$a \geq m \quad \text{for any } a \in A.$$

(iii) <u>bounded</u>, if it is bounded both from above and below.

In the previous definitions, the numbers M, m are called <u>upper</u> and <u>lower bounds</u> for the set A, respectively.

<u>PROPOSITION 2.1</u>: The set $A \subseteq \mathbb{R}$ is bounded if and only if there exists some $M > 0$ such that
$$|a| \leq M \quad \text{for all } a \in A.$$

PROPOSITION 2.1: The set $A \subseteq \mathbb{R}$ is bounded if and only if there exists some $M > 0$ such that

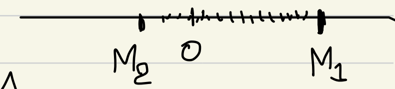
$$|a| \leq M \quad \text{for all } a \in A.$$

PROOF

$\Rightarrow$: Assume A is bounded.

There exist $M_1, M_2 \in \mathbb{R}$ such that

$$M_2 \leq a \leq M_1, \quad \text{for all } a \in A.$$



Set $M = \max\{|M_1|, |M_2|\}$, then: