PROPOSITION 5.12 (Partial Integration):

If $f, g : [a,b] \to \mathbb{R}$ are differentiable on $[a,b]$ and $f', g' : [a,b] \to \mathbb{R}$ are Riemann-integrable, then

$$\int_a^b f'(x) g(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f(x) g'(x)\,dx.$$

Examples:

(i) $$\int_1^e \ln x\,dx = \int_1^e (x)' \ln x\,dx$$

$$= [x \ln x]_1^e - \int_1^e x \cdot (\ln x)'\,dx$$

$$= e \ln e - \ln 1 - \int_1^e x \cdot \frac{1}{x}\,dx$$

$$= e - \int_1^e 1\,dx$$

$$= e - (e-1)$$

$$= 1.$$

(ii) $$\int_0^1 x^2 e^x\,dx = \int_0^1 x^2 (e^x)'\,dx$$

$$= [x^2 e^x]_0^1 - \int_0^1 2x e^x\,dx$$

$$= e - \int_0^1 2x (e^x)'\,dx$$

$$= e - \left[2xe^x\right]_0^1 + \int_0^1 2e^x\,dx$$

$$= e - 2e + \left[2e^x\right]_0^1$$

$$= e - \cancel{2}e + \cancel{2}e - 2$$

$$= e - 2.$$

(iii) Find $I = \int_0^{\pi} e^{2x}\cos x\,dx$

$$I = \int_0^{\pi} e^{2x}(\sin x)'\,dx$$

$$= \cancel{\left[e^{2x}\sin x\right]_0^{\pi}} - \int_0^{\pi} 2e^{2x}\sin x\,dx$$

$$= \int_0^{\pi} 2e^{2x}(\cos x)'\,dx$$

$$= \left[2e^{2x}\cos x\right]_0^{\pi} - \int_0^{\pi} 4e^{2x}\cos x\,dx$$

$$= 2e^{2\pi}\cos\pi - 2 - 4I$$

$$= -2e^{2\pi} - 2 - 4I \Rightarrow$$

$$5I = -2(e^{2\pi}+1) \Rightarrow$$

$$I = -\frac{2}{5}(e^{2\pi}+1).$$

PROPOSITION 5.13: Suppose $g:[a,b]\to\mathbb{R}$ has a continuous derivative and f is continuous on some interval which contains $g([a,b])$. Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt.$$

In partice, for $\int_a^b f(g(x))\,g'(x)\,dx$:

we set $u = g(x)$

$$du = g'(x)\,dx.$$

$x_1 = a \Rightarrow u_1 = g(a)$

$x_2 = b \Rightarrow u_2 = g(b)$.

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du = \int_{g(a)}^{g(b)} f(y)\,dy$$

$$= \int_{g(a)}^{g(b)} f(t)\,dt \ldots$$

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^2 2e^u\,du = \left[2e^u\right]_1^2 = 2(e^2-e).$$

We set $u=\sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\,dx.$$

$x_1 = 1 \Rightarrow u_1 = 1$

$x_2 = 4 \Rightarrow u_2 = 2$

$$\int_1^{\sqrt{6}} x\sqrt{3+x^2}\,dx =$$

Set $u = 3 + x^2$

$du = 2x\,dx$

$x_1 = 1 \Rightarrow u_1 = 4$

$x_2 = \sqrt{6} \Rightarrow u_2 = 9$

$$= \int_4^9 \frac{\sqrt{u}}{2}\,du = \left[\frac{1}{\cancel{2}}\,\frac{u^{3/2}}{\frac{3}{\cancel{2}}}\right]_4^9$$

$$= \left[\frac{\sqrt{u}^{\,3}}{3}\right]_4^9 = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}.$$

- Find the area among the graph of $f(x) = \dfrac{1}{x^2+2x+5}$, the horizontal axis and the lines $x=-1$, $x=1$.

The area we want is

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x+1)^2+4}$$

Set $u = x+1$

$du = dx$

$x_1 = -1 \Rightarrow u_1 = 0$

$x_2 = 1 \Rightarrow u_2 = 2$

$$= \int_0^2 \frac{du}{u^2+4}$$

Set $u = 2t$

$du = 2\,dt$

$u_1 = 0 \Rightarrow t_1 = 0$

$u_2 = 2 \Rightarrow t_2 = 1$

$$= \int_0^1 \frac{2\,dt}{4(t^2+1)}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{t^2+1} = \frac{1}{2}\left[\arctan t\right]_0^1$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}.$$

A function $f:[a,b]\to\mathbb{R}$ is called <u>piecewise continuous</u> if there exist $a=x_0<x_1<x_2<\dots<x_n=b$ and continuous functions

$$g_i:[x_{i-1},x_i]\to\mathbb{R},\qquad i=1,2,\dots,n$$

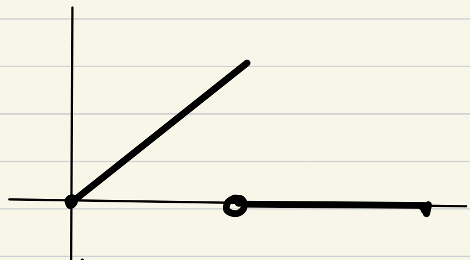
such that for $i=1,2,\dots,n$

$$f(x)=g_i(x)\quad\text{for all } x\in[x_{i-1},x_i]$$

<u>Examples</u>

(i) $f:[0,2]\to\mathbb{R}$,

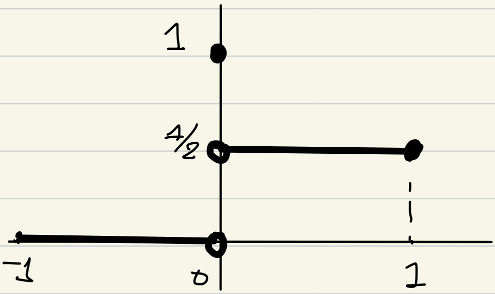
$$f(x)=\begin{cases} x, & 0\le x\le 1\\ 0, & 1<x\le 2\end{cases}$$



Then f is piecewise continuous on $[0,2]$.

(ii) $g:[-1,1]\to\mathbb{R}$

$$g(x)=\begin{cases} 0, & -1\le x<0\\ 1, & x=0\\ \frac{1}{2}, & 0<x\le 1\end{cases}$$

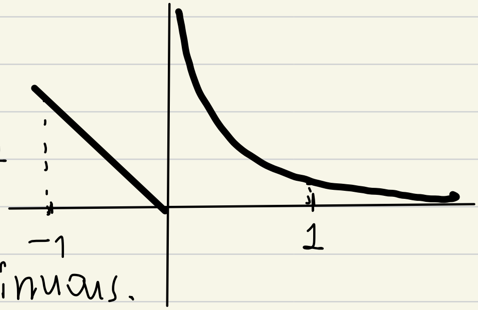


g is piecewise continuous function on $[-1,1]$.

According to the def.: consider the intervals $[-1,0]$, $[0,1]$ and $G_1:[-1,0]\to\mathbb{R}$, $G_1(x)=0$ and $G_2:[0,1]\to\mathbb{R}$, $G_2(x)=\frac{1}{2}$.

(iii) $h : [-1, 1] \to \mathbb{R}$,

$$h(x) = \begin{cases} -x, & -1 \leq x \leq 0 \\ \frac{1}{x}, & 0 < x \leq 1 \end{cases}$$



h is NOT piecewise continuous.

This is because there does not exist $H : [0, 1] \to \mathbb{R}$ which is continuous and satisfies

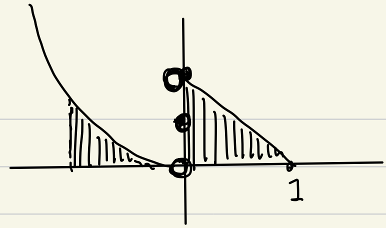
$$H(x) = h(x), \quad x \in [0, 1].$$

THEOREM 5.14: Let $f : [a, b] \to \mathbb{R}$ be a piecewise continuous function on $[a, b]$. Then f is Riemann integrable and

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} g_i(x)\,dx,$$

where $x_0, x_1, \dots, x_n$ and $g_1, g_2, \dots, g_n$ are as in the definition.

E.g. $f(x) = \begin{cases} x^2, & x < 0 \\ 1/2, & x = 0 \\ 1-x, & x > 0 \end{cases}$



$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} x^2\,dx + \int_{0}^{1} (1-x)\,dx$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

- For the function $\operatorname{sgn}(x) = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

$$\int_{-1}^{2} \operatorname{sgn}(x)\,dx = \int_{-1}^{0} \operatorname{sgn}(x)\,dx + \int_{0}^{2} \operatorname{syn}(x)\,dx$$

$$= -1 + 2 = 1.$$

Let $f : D \to \mathbb{R}$ be a function, and $D \subseteq \mathbb{R}$ be "symmetric around 0", i.e.

$$\forall x \in \mathbb{R} : \quad x \in D \iff -x \in D.$$

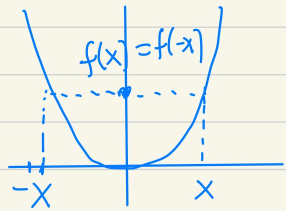
We say $f : D \to \mathbb{R}$ is :

- <u>even</u>, if for all $x \in D$
$$f(-x) = f(x)$$
- <u>odd</u>, if for all $x \in D$
$$f(-x) = -f(x).$$

E.g. $f(x) = x^2$, $x \in \mathbb{R}$

$f(-x) = (-x)^2 = x^2 = f(x)$

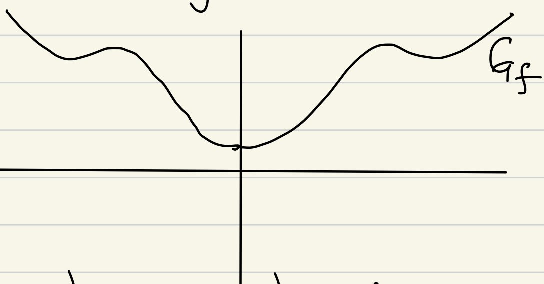
f is an even function



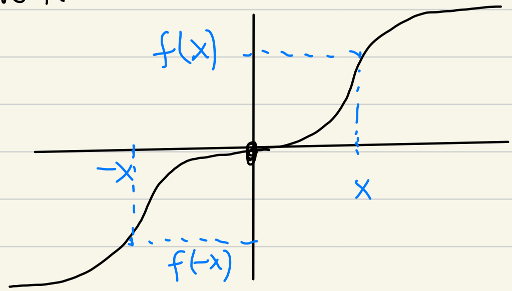
$g(x) = x^5$, $x \in \mathbb{R}$

$g(-x) = (-x)^5 = -x^5 = -g(x)$

g is an odd function.



The graph of any even function is symmetric around the axis $x = 0$.

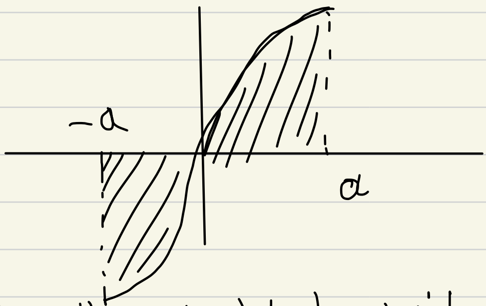


The graph of any odd function is symmetric around the origin $O(0,0)$.

Suppose f is Riemann-integrable.
If f is odd and $a>0$,
then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$



In the first integral in the right-hand-side, we set

$$u = -x$$
$$du = -dx$$
$$x_1 = -a \Rightarrow u_1 = a$$
$$x_2 = 0 \Rightarrow u_2 = 0$$

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-u)\,du$$

$$= \int_{0}^{a} f(-u)\,du = -\int_{0}^{a} f(u)\,du$$

$$= -\int_{0}^{a} f(x)\,dx$$

Thus

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Similarly, if f is even and R.-int. then for any $a>0$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

E.g. We can find $\int_{-1}^{1} x^{10}\sin(x^7)\,dx$

The function

$$f(x) = x^{10}\sin(x^7), \quad x\in\mathbb{R}$$

is odd:

$$f(-x) = (-x)^{10}\sin((-x)^7) = -f(x).$$

Therefore

$$\int_{-1}^{1} x^{10}\sin(x^7)\,dx = 0.$$

If $f : [a,b] \to \mathbb{R}$ is Riemann-integrable, the real number

$$\frac{1}{b-a} \int_a^b f(x)\, dx$$

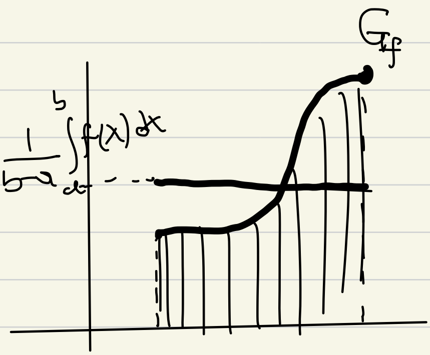
is called the <u>mean value</u> of f on the interval $[a,b]$.

Suppose $g : [a,b] \to \mathbb{R}$ is constant and has the property

$$\int_a^b g(x)\, dx = \int_a^b f(x)\, dX.$$

Then

$$g(x) = \frac{1}{b-a} \int_a^b f(x) dx.$$



THEOREM 5.15 (Mean Value Theorem of Integral Calculus): If $f : [a, b] \to \mathbb{R}$ is continuous, there exists $\xi \in (a, b)$ such that

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

PROOF