THEOREM 3.5 (Mean Value Theorem, Lagrange):
Let $f : [a,b] \to \mathbb{R}$. Then if
- f is continuous on $[a,b]$
- f is differentiable on (a,b)

then there exists $x_0 \in (a,b)$ with
$$f'(x_0) = \frac{f(b)-f(a)}{b-a}.$$

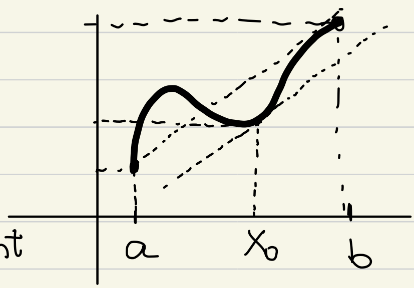
PROOF

Set $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$.

Then g is continuous on $[a,b]$ and differentiable on (a,b) with $g(a) = g(b) = f(a)$.

So by Rolle's theorem there exists $x_0 \in (a,b)$ with
$$g'(x_0) = 0 \Rightarrow f'(x_0) = \frac{f(b)-f(a)}{b-a}. \quad \blacksquare$$

Geometrically, the MTV states that there exists at least one point $(x_0, f(x_0))$ where the tangent line at the graph of f is parallel to the segment from $A(a, f(a))$ to $B(b, f(b))$.



Let $f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ an interval.
The function f is called:

(i) <u>increasing</u> on I, if for all $x, y \in I$
$$x < y \Rightarrow f(x) \leq f(y).$$

(ii) <u>strictly increasing</u> on I, if for all $x, y \in I$
$$x < y \Rightarrow f(x) < f(y).$$

(iii) <u>decreasing</u> on I, if for all $x, y \in I$
$$x < y \Rightarrow f(x) \geq f(y)$$

(iv) <u>Strictly decreasing</u> on I, if for all $x, y \in I$
$$x < y \Rightarrow f(x) > f(y).$$

Also f is called <u>monotonic</u> or <u>strictly monotonic</u> if it is either decreasing or increasing (and strictly incr. or strictly decr., respectively).

* Sometimes f is called non-decreasing instead of increasing, and also increasing instead of strictly increasing.

THEOREM 3.6: Let $f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ an interval, be differentiable in the interior of I.

(i) If $f'(x) > 0$ for all x, then f is strictly increasing on I.

(ii) If $f'(x) \geq 0$ for all x (in the int. of I) then f is increasing on I.

(iii) If $f'(x) < 0$ for all x, then f is strictly decreasing on I.

(iv) If $f'(x) \leq 0$ for all x, then f is decreasing on I.

PROOF

(i) Take $x, y \in I$ with $x < y$.
We need to prove that $f(x) < f(y)$.
By the Mean Value Theorem, there exists $x_0 \in (x, y)$ with

$$f'(x_0) = \frac{f(y) - f(x)}{y - x}.$$

By the hypothesis,

$$f'(x_0) > 0 \Rightarrow \frac{f(y) - f(x)}{y - x} > 0 \Rightarrow f(x) < f(y).$$

(ii), (iii), (iv): Similar. ■

* A constant function $f(x) = c$ $(x \in I)$ is both increasing and decreasing but neither strictly incr. nor strictly dec.

Example: $f(x) = \dfrac{e^x}{x}$, $x \in (0, \infty)$

Then $f'(x) = \dfrac{xe^x - e^x}{x^2} = \dfrac{x-1}{x^2} e^x$.

$f'(x) = 0 \iff x = 1$
$f'(x) > 0 \iff x > 1$
$f'(x) < 0 \iff x < 1$.

x	0		1		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$		$\searrow$		$\nearrow$	

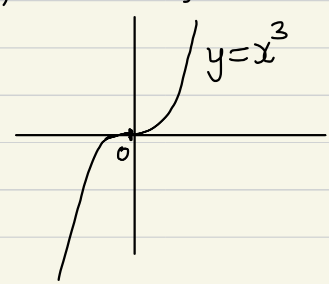
f is strictly decreasing in $(0,1]$ and strictly increasing in $[1,\infty)$.

f has a local minimum at $x=1$, which is $f(1) = e$.
This is also a global minimum.

The converse to Theorem 3.6 is not true, i.e. if f is strictly increasing, this does not necessarily imply that

$$f'(x) > 0 \quad \text{for all } x.$$

Take for example $f(x) = x^3$.
f is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$.



If $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$, does that automatically imply that f is increasing on some interval around x_0?

The answer is NO. Take for example

$$f(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f is differentiable on $\mathbb{R}$, with

$$f'(x) = \begin{cases} 1 + 4x \sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 1, & x = 0 \end{cases}$$

We have $f'(0) = 1 > 0$
but f is not increasing on any interval of the form $(-\delta, \delta)$.

<u>THEOREM 3.7</u>: Let $f: I \to \mathbb{R}$ be continuous on the interval I and differentiable in the interior of I.
If $f'(x) = 0$ for all x in the int. of I then $f(x) = C$ for all $x \in I$.

<u>PROOF</u>

Let $x, y \in I$ with $x < y$.
By the Mean Value Theorem,
$\exists\, t \in (x, y)$ such that $f'(t) = \dfrac{f(y) - f(x)}{y - x}$.

But $f'(t)=0 \Rightarrow \frac{f(y)-f(x)}{y-x}=0 \Rightarrow f(x)=f(y)$. ∎

* In a previous counterexample, we used the function

$$f(x)=\begin{cases} x+2x^2\sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0. \end{cases}$$

How did we calculate $f'(0)$?

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x+2x^2\sin\left(\frac{1}{x}\right)}{x}$$

$$=\lim_{x\to 0}\left(1+2x\sin\left(\frac{1}{x}\right)\right)=1$$

because $\left|x\sin\left(\frac{1}{x}\right)\right|\leq |x| \Rightarrow$

$$-|x|\leq x\sin\left(\frac{1}{x}\right)\leq |x|$$

and $\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right]=0$.

Therefore this means that f is diff. at 0 with $f'(0)=1$.

COROLLARY 3.8: Let $f, g : I \to \mathbb{R}$ (I an interval) be continuous and

$$f'(x) = g'(x) \quad \text{for all } x \text{ in the int. of } I.$$

Then $f(x) = g(x) + c$ for all $x \in I$ (where $c \in \mathbb{R}$ is a constant.)

Example: Prove that $\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arctan x$, $x > 0$.

• Set $h(x) = \arctan\left(\frac{1}{x}\right) + \arctan x$, $x \in (0, \infty)$.

(I choose h that way because we actually have to show that this function is constant and equal to $\frac{\pi}{2}$).

Then
$$h'(x) = \frac{1}{1 + \left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{1 + x^2} = 0$$

so $h(x) = c$ for all $x > 0$.

We can evaluate the constant c:

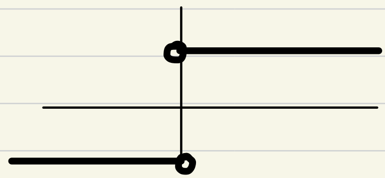
$$c = h(1) = 2\arctan 1 = \frac{\pi}{2}.$$

So, for all $x > 0$: $h(x) = \frac{\pi}{2} \Rightarrow$
$$\arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arctan x.$$

REMARK: Theorem 3.7 can only be applied on intervals.

Consider the function $g: \mathbb{R}\setminus\{0\} \to \mathbb{R}$,

$$g(x) = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}.$$



Then $g'(x)=0$ for all $x\in(-\infty,0)\cup(0,\infty)$ but g is not equal to a constant on $\mathbb{R}\setminus\{0\}$.

This is because $\mathbb{R}\setminus\{0\}$ is not an interval and we cannot apply Thm 3.7.

THEOREM 3.9 (Generalised Mean Value Theorem, Cauchy): Let $f,g:[a,b]\to\mathbb{R}$. If:

- f, g are cont. on $[a,b]$
- f, g are diff. in (a,b)
- $g'(x)\neq 0$ for all $x\in(a,b)$

then there exists $x_0\in(a,b)$ such that,

$$\frac{f'(x_0)}{g'(x_0)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

PROOF

Apply Rolle's Theorem to $h:[a,b]\to\mathbb{R}$,

$$h(x) = (g(b)-g(a))f(x) - (f(b)-f(a))g(x).$$

THEOREM 3.10 (de l'Hospital): If f, g are differentiable on some interval around x_0 (x_0 is real or $\pm\infty$) and

- $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$
- $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$ (L is real or $\pm\infty$)

then also $\lim_{x \to x_0} \frac{f(x)}{g(x)} = L$.

(This applies to both "regular" limits and also side limits).

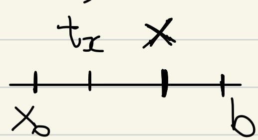
PROOF

- We first deal with the case of a right side-limit.

Suppose f, g are diff. on $(x_0, b]$.
Define

$$F(x) = \begin{cases} f(x), & x_0 < x \le b \\ 0, & x = x_0 \end{cases}, \quad G(x) = \begin{cases} g(x), & x_0 < x \le b \\ 0, & x = x_0 \end{cases}.$$

Let $x_0 < x < b$.



Then applying Theorem 3.9 to F, G on $[x_0, x]$ — note that they are both continuous) — there exists some
$t = t_x \in (x_0, x)$ with

$$\frac{F'(t_x)}{G'(t_x)} = \frac{F(x)-F(x_0)}{G(x)-G(x_0)} \Rightarrow \frac{f'(t_x)}{g'(t_x)} = \frac{f(x)}{g(x)}.$$

When $x \to x_0^+$, then also $t_x \to x_0$.
Thus

$$\lim_{x\to x_0^+} \frac{f'(t_x)}{g'(t_x)} = \lim_{x\to x_0} \frac{f(x)}{g(x)} \Rightarrow$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L.$$

- For the case of a left-side limit we work similarly.
- The case of a two-sided limit follows from the side limits.
- It remains to deal with the case when $x_0 = \pm\infty$.

Set $f_1(x) = f\left(\frac{1}{x}\right)$, $g_1(x) = g\left(\frac{1}{x}\right)$, $0 < x < a$.

Apply the proof for the right side-limits at 0 and observe that

$$\lim_{x\to 0^+} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to 0^+} \frac{f'\left(\frac{1}{x}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{g'\left(\frac{1}{x}\right)\cdot\cancel{\left(-\frac{1}{x^2}\right)}} = \lim_{y\to\pm\infty} \frac{f'(y)}{g'(y)} = L. \blacksquare$$

The same conclusion is also true when dealing with indefinite limits of the form $\frac{\pm\infty}{\pm\infty}$ (proof omitted).

E.g. Find $\lim_{x\to 0^+} x^{\sqrt{x}}$.

$$x^{\sqrt{x}} = e^{\log x^{\sqrt{x}}} = e^{\sqrt{x}\log x}.$$

Now $\lim_{x\to 0^+} \sqrt{x}\log x = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{\sqrt{x}}}$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2\sqrt{x}\cdot x}} = 0$$

So $\lim_{x\to 0^+} x^{\sqrt{x}} = e^0 = 1$.