THEOREM 5b: If $f:[a,b]\to\mathbb{R}$ is continuous, then it is Riemann integrable.

PROOF

Since $f:[a,b]\to\mathbb{R}$ is continuous it is also uniformly continuous.

Let $\varepsilon>0$. There exists some $\delta=\delta(\varepsilon)>0$ such that

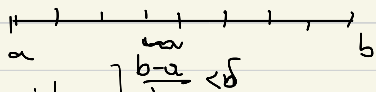
$$|x-y|<\delta \quad \text{implies} \quad |f(x)-f(y)|<\frac{\varepsilon}{b-a}.$$

Let $n\geq 1$ be such that

$$\frac{b-a}{n}<\delta$$

and consider the partition

$$P_n=\left\{a,\ a+\frac{b-a}{n},\ a+\frac{2(b-a)}{n},\ \dots,\ b\right\}.$$



Set

$$m_j=\min\left\{f(t):\ a+(j-1)\frac{b-a}{n}\leq t<a+j\frac{b-a}{n}\right\},$$

$$M_j=\max\left\{f(t):\ a+(j-1)\frac{b-a}{n}\leq t<a+j\frac{b-a}{n}\right\}$$

$(j=1,2,\dots,n)$.

For each $j=1,2,\dots,n$ we have

$$m_j=f(x_j) \quad \text{and} \quad M_j=f(y_j)$$

for some x_j,y_j with

$$|x_j-y_j|<\frac{b-a}{n}<\delta,$$

hence $|M_j-m_j|=|f(x_j)-f(y_j)|<\frac{\varepsilon}{b-a}$.

Now

$$U(f, P_n) - L(f, P_n) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}) - \sum_{j=1}^{n} m_j (x_j - x_{j-1})$$

$$= \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1})$$

$$< \sum_{j=1}^{n} \frac{\varepsilon}{b-a} (x_j - x_{j-1})$$

$$= \frac{\varepsilon}{b-a} \sum_{j=1}^{n} (x_j - x_{j-1})$$

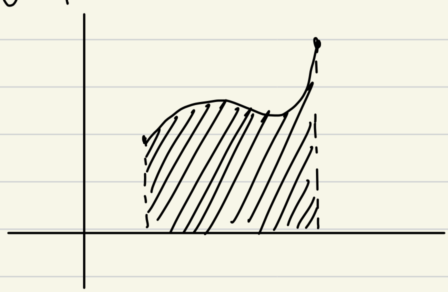
$$= \varepsilon .$$

Therefore f is Riemann integrable. ■

If $f(x) \geq 0$ on $[a, b]$ and f is Riemann integrable on $[a, b]$ then

$\int_a^b f(x)\,dx$ is the area between

the lines $x = a$, $x = b$, the graph of f and the horizontal axis.



Whenever f is Riemann integrable on a closed and bounded interval I and $a, b \in I$ we define:

- $$\int_a^a f(x)\,dx = 0$$
- $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

PROPERTIES OF RIEMANN INTEGRATION

1. If f is Riemann-integrable on the closed bounded interval I and $a, b, c \in I$ then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

2. If $f, g : [a,b] \to \mathbb{R}$ are Riemann-integrable and $\lambda, \mu \in \mathbb{R}$, then the function $\lambda f + \mu g$ is also Riemann-integrable and

$$\int_a^b (\lambda f(x) + \mu g(x))\,dx = \lambda \int_a^b f(x)\,dx + \mu \int_a^b g(x)\,dx$$

3. If $f:[a,b]\to\mathbb{R}$ is Riemann-integrable then so is $|f|$, and

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx .$$

(Triangle Inequality for integrals).

4. If $f,g:[a,b]\to\mathbb{R}$ are both Riemann-integrable then $f\cdot g$ is also Riemann-integrable, i.e.

$$\int_a^b f(x)g(x)\,dx$$

is well-defined.

5. If $f,g:[a,b]\to\mathbb{R}$ are Riemann-integrable on $[a,b]$ and

$$f(x)\leq g(x) \quad \text{for all } x\in[a,b]$$

then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx .$$

As a special case of this, if $f:[a,b]\to\mathbb{R}$ is Riemann integrable with $m\leq f(x)\leq M$, $x\in[a,b]$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

In the proof of Theorem 5.6 we used the specific partition

$$P_n = \left\{ a + k\frac{b-a}{n} : k = 0, 1, \dots, n \right\}$$

of $[a,b]$, i.e. we split $[a,b]$ into n subintervals of equal length. This partition can be used to prove the following fact:

PROPOSITION 5.7: Assume f is Riemann integrable on $[a,b]$. Then

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a+k\frac{b-a}{n}\right) = \int_a^b f(x)\,dx.$$

COROLLARY 5.8: If $f:[0,1]\to\mathbb{R}$ is Riemann integrable, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx.$$

We can calculate $\int_a^b x\,dx$ using Proposition 5.7.

$$\int_a^b x\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n}\left(a+k\cdot\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left(na + \frac{b-a}{n}\sum_{k=1}^{n} k\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left(na + \frac{b-a}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)}{2}\right)$$

$$= \lim_{n\to\infty}\left(ba-a^2 + \frac{(b-a)^2}{2}\cdot\frac{n+1}{n}\right) = \frac{b^2-a^2}{2}.$$

THEOREM 5.9 (Fundamental Theorem of Calculus): Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable and $x_0 \in [a,b]$. Define

$$F(x) = \int_{x_0}^{x} f(t)\,dt \;, \qquad x\in[a,b].$$

If f is continuous at the point $x\in[a,b]$, then F is differentiable at x with $F'(x) = f(x)$.

PROOF

We will consider the case $x\in(a,b)$ (the proof when $x=a$ or $x=b$ is similar). For $h\neq 0$ sufficiently small,

$$\frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\left(\int_{x_0}^{x+h} f(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right) - f(x)$$

$$= \frac{1}{h}\int_{x}^{x+h} f(t)\,dt - f(x)$$

$$= \frac{1}{h}\int_{x}^{x+h} f(t)\,dt - \frac{1}{h}\int_{x}^{x+h} f(x)\,dt$$

$$= \frac{1}{h}\int_{x}^{x+h} (f(t)-f(x))\,dt.$$

Let $\varepsilon > 0$. Since f is continuous at x, there exists $\delta > 0$ such that

$$|t - x| < \delta \implies |f(t) - f(x)| < \varepsilon.$$

So for $|h| < \delta$,

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\, dt \right|$$

$$= \frac{1}{|h|} \left| \int_x^{x+h} [f(t) - f(x)]\, dt \right|$$

$$< \frac{1}{|h|} |h| \varepsilon = \varepsilon.$$

We have shown that

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x). \quad \blacksquare$$

As an immediate corollary:

THEOREM 5.10: Suppose $f:[a,b]\to\mathbb{R}$ is continuous and $x_0\in[a,b]$. Then

$$F(x) = \int_{x_0}^{x} f(t)\,dt \,, \qquad x\in[a,b]$$

is an antiderivative of f in $[a,b]$.

This means that for any continuous function $f:[a,b]\to\mathbb{R}$, we automatically know an antiderivative, the function

$$F(x) = \int_{a}^{x} f(t)\,dt \,, \qquad x\in[a,b].$$

This observation allows us to calculate Riemann integrals using antiderivatives.

PROPOSITION 5.11: Assume $f:[a,b]\to\mathbb{R}$ is continuous and F is an antiderivative of f on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF

Let $G(x) = \int_{x_0}^{x} f(t)\,dt$, $x\in[a,b]$

where $x_0 \in [a,b]$. Then G is an antiderivative of f, so

$$G(x) = F(x) + c, \quad x\in[a,b]$$

for some constant $c\in\mathbb{R}$. Hence

$$\begin{aligned}\int_a^b f(x)\,dx &= G(b) - G(a)\\ &= (F(b)+c) - (F(a)+c)\\ &= F(b) - F(a). \qquad \blacksquare\end{aligned}$$

For convenience, we write

$$F(b) - F(a) = \big[F(x)\big]_a^b = F(x)\Big|_a^b = F(x)\Big|_{x=a}^{x=b}$$

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\int_0^9 2\sqrt{x}\,dx = \int_0^9 2x^{\frac{1}{2}}\,dx = \left[2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9 = 36.$$

$$\int_1^e \frac{dx}{x} = \left[\ln x\right]_1^e = \ln e - \ln 1 = 1.$$

$$\int_3^4 \frac{dx}{x^2-3x+2} = \int_3^4 \frac{dx}{(x-1)(x-2)}$$

$$= \int_3^4 \left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx$$

$$= \left[\ln|x-2| - \ln|x-1|\right]_3^4$$

$$= \left[\ln\left|\frac{x-2}{x-1}\right|\right]_3^4 = \ln\frac{4}{3}.$$