MA0301 ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2017

Exam 1

Exercise 1: 10 points
Exercise 2: 20 points
Exercise 3: 15 points
Exercise 4: 20 points
Exercise 5: 15 points
Exercise 6: 10 points
Exercise 7: 10 points

Total: 100 points

Exercise 1. Logic (10 points) *Use the laws of logic*

(1) **(5 points)** *to simplify:*

$$\big(p \wedge (\neg s \vee q \vee \neg q)\big) \vee \big((s \vee t \vee \neg s) \wedge \neg q\big)$$

(2) **(5 points)** *to show that:*

$$\big(p \rightarrow (q \vee r)\big) \Leftrightarrow \big((p \wedge \neg q) \rightarrow r\big)$$

Solution 1. (1) We want to simplify $\big(p \wedge (\neg s \vee q \vee \neg q)\big) \vee \big((s \vee t \vee \neg s) \wedge \neg q\big)$.

$$\begin{aligned}
\big(p \wedge (\neg s \vee q \vee \neg q)\big) \vee \big((s \vee t \vee \neg s) \wedge \neg q\big) &\Leftrightarrow \big(p \wedge (\neg s \vee (q \vee \neg q))\big) \vee \big((t \vee (s \vee \neg s)) \wedge \neg q\big)\\
&\Leftrightarrow \big(p \wedge (\neg s \vee T)\big) \vee \big((t \vee T) \wedge \neg q\big)\\
&\Leftrightarrow (p \wedge T) \vee (T \wedge \neg q)\\
&\Leftrightarrow p \vee \neg q.
\end{aligned}$$

We used that $q \vee p = p \vee q$ and that, in general, $q \vee \neg q = T$, where T is the truth value.

(2) We want to show that $\big(p \rightarrow (q \vee r)\big) \Leftrightarrow \big((p \wedge \neg q) \rightarrow r\big)$.

$$\begin{aligned}
\big(p \rightarrow (q \vee r)\big) &\Leftrightarrow \neg p \vee (q \vee r)\\
&\Leftrightarrow (\neg p \vee q) \vee r\\
&\Leftrightarrow [\neg\neg(\neg p \vee q)] \vee r\\
&\Leftrightarrow [\neg(p \wedge \neg q)] \vee r\\
&\Leftrightarrow \big((p \wedge \neg q) \rightarrow r\big).
\end{aligned}$$

Date: June 10, 2017.

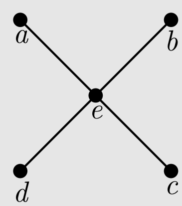
We used logical equivalence, $p \to q \Leftrightarrow (\neg p \vee q)$; associativity; double negation; De Morgan's law; double negation; and logical equivalence.

Exercise 2. Partially ordered sets (20 points)

(1) **(3 points)** *Write down the definition of a partial order.*

(2) **(5 points)** *Does the relation* $R := \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,c),(c,d),(d,a)\}$ *define a partial ordering on* $A := \{a,b,c,d\}$*?*

(3) **(5 points)** *List the set* A *and express the relation* R *as a set of ordered pairs for the Hasse diagram*



(4) **(7 points)** *Construct the Hasse diagram of the partially ordered set* (A,R)*, where* $A := \{a,b,c,d\}$ *and* $R := \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,c),(a,c),(c,d),(a,d),(b,d)\}$.

Solution 2. (1) A relation R on a set A is called a partial order of A if R is reflexive, anti-symmetric and transitive. A together with a partial order R is called a partially ordered set (poset).

(2) No.

(3) $A = \{a,b,c,d,e\}$ and $R = \{(a,a),(b,b),(c,c),(d,d),(e,e),(d,b),(d,a),(d,e),(c,e),(c,a),(c,b),(e,a),(e,b)\}$.

(4)

Exercise 3. Boolean algebra (15 points) *Use Boolean algebra to*

(1) **(6 points)** *prove that*

$$x'y' + x'y + xy = x' + y$$

(2) **(6 points)** *prove that*

$$y + x'z + xy' = x + y + z$$

(3) **(3 points)** *simplify*

$$xyz + xyz' + x'y$$

Solution 3. (1) We want to show that $x'y' + x'y + xy = x' + y$.

$$\begin{aligned} x'y' + x'y + xy &= x'y' + x'y + x'y + xy \\ &= x'(y + y') + y(x' + x) \\ &= x' + y. \end{aligned}$$

In the first line we used the idempotent law $x'y+x'y = x'y$. In the second line we used that $x'+x = y'+y = 1$.

(2) We want to show that $y + x'z + xy' = x + y + z$.

$$\begin{aligned} y + x'z + xy' &= y(1 + x) + x'z + xy' \\ &= (y + x)(y' + y) + x'z \\ &= y + x + x'z \\ &= y + (x + x')(x + z) \\ &= y + x + z \end{aligned}$$

In the first line we used the null law, $1 + x = 1$. In the second line we used $yy = y$ and $yy' = 0$. In the third line we used $y + y' = 1$. In the fourth line we used again $xx = x$ and $x'x = 0$.

(3) We want to simplify $xyz + xyz' + x'y$.

$$xyz + xyz' + x'y = xy(z + z') + x'y = xy + x'y = (x + x')y = y.$$

Exercise 4. Induction (20 points)

(1) **(4 points)** *Prove that*

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4}.$$

(2) *Define* $\bigcup_{i=1}^{n} A_i$ *by* $\bigcup_{i=1}^{1} A_i = A_1$ *and* $\bigcup_{i=1}^{n+1} A_i = \left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}$. *Define* $\bigcap_{i=1}^{n} A_i$ *by* $\bigcap_{i=1}^{1} A_i = A_1$ *and* $\bigcap_{i=1}^{n+1} A_i = \left(\bigcap_{i=1}^{n} A_i\right) \cap A_{n+1}$.

i) **(4 points)** *Prove that*

$$\left(\bigcup_{i=1}^{n} A_i\right) \cap A = \bigcup_{i=1}^{n} (A_i \cap A).$$

ii) **(4 points)** *Prove that*

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \overline{A_i}.$$

Recall that $\overline{A}$ denotes the complement of A.

(3) **(8 points)** *Use the trigonometrical addition formulas:*

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

to prove that for n a positive integer

$$\big(\cos(c) + i\sin(c)\big)^n = \cos(nc) + i\sin(nc).$$

This formula is known as De Moivre's Theorem. Note that i is the so-called imaginary unit, and you should use that $i^2 = -1$.

Solution 4. (1) Base step $n = 1$: $1^3 = 1^2 2^2/4$, the statement is true for $n = 1$.

Ind. step: assume it is true for $n = k$, i.e., $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$. We want to prove it for $n = k+1$:

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4}.\end{aligned}$$

This shows that the statement holds for $n = k+1$.

(2.i) Base step $n = 1$: $A_1 \cap A = A_1 \cap A$, the statement is true for $n = 1$.

Ind. step: assume it is true for $n = k$, $\left(\bigcup_{i=1}^{k} A_i\right) \cap A = \bigcup_{i=1}^{k}(A_i \cap A)$. We want to show it for $n = k+1$.

$$\begin{aligned}\left(\bigcup_{i=1}^{k+1} A_i\right) \cap A &= \left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right) \cap A \\ &= (\left(\bigcup_{i=1}^{k} A_i\right) \cap A) \cup (A_{k+1} \cap A) \\ &= \bigcup_{i=1}^{k}(A_i \cap A) \cup (A_{k+1} \cap A) \\ &= \bigcup_{i=1}^{k+1}(A_i \cap A)\end{aligned}$$

This shows that the statement holds for $n = k+1$.

(2.ii) Base step $n = 1$: $\overline{A_1} = \overline{A_1}$, the statement is true for $n = 1$.

Ind. step: assume it is true for $n = k$, $\overline{\left(\bigcup_{i=1}^k A_i\right)} = \bigcap_{i=1}^k \overline{A_i}$. We want to show it for $n = k + 1$.

$$\begin{aligned}\overline{\left(\bigcup_{i=1}^{k+1} A_i\right)} &= \overline{\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)} \\ &= \overline{\left(\bigcup_{i=1}^{k} A_i\right)} \cap \overline{A_{k+1}} \\ &= \bigcap_{i=1}^{k} \overline{A_i} \cap \overline{A_{k+1}} = \bigcap_{i=1}^{k+1} \overline{A_i}.\end{aligned}$$

This shows that the statement holds for $n = k + 1$.

(3) Base step $n = 1$: $\big(\cos(c) + i\sin(c)\big)^1 = \cos(1c) + i\sin(1c)$, the statement is true for $n = 1$.

Ind. step: assume it is true for $n = k$, $\big(\cos(c) + i\sin(c)\big)^k = \cos(kc) + i\sin(kc)$. We want to show it for $n = k + 1$.

$$\begin{aligned}\big(\cos(c) + i\sin(c)\big)^{k+1} &= \big(\cos(c) + i\sin(c)\big)^k\big(\cos(c) + i\sin(c)\big) \\ &= \big(\cos(kc) + i\sin(kc)\big)\big(\cos(c) + i\sin(c)\big) \\ &= \cos(kc)\cos(c) - \sin(kc)\sin(c) + i\big(\sin(kc)\cos(c) + \cos(kc)\sin(c)\big) \\ &= \cos((k+1)c) + i\sin((k+1)c).\end{aligned}$$

This shows that the statement holds for $n = k + 1$.

Exercise 5. Binomial coefficients (15 points)

(1) **(2 points)** *Use the binomial theorem to compute*

$$\sum_{i=0}^{27} \binom{27}{i} (-3)^{2i+1}$$

(2) **(10 points)** *Use the binomial theorem to show Vandermonde's formula*

$$\binom{a+b}{r} = \sum_{k=0}^{r} \binom{a}{k}\binom{b}{r-k}, \tag{1}$$

where a, b, f are positive integers and $r \le \min(a, b)$.

(3) **(3 points)** *Use Vandermonde's formula* (1) *to show that*

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{k}$$

Solution 5. (1) The binomial theorem says that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Hence

$$(x+1)^{27} = \sum_{k=0}^{27} \binom{27}{k} x^k.$$

Note that

$$\sum_{i=0}^{27} \binom{27}{i} (-3)^{2i+1} = -3 \sum_{i=0}^{27} \binom{27}{i} (-3)^{2i} = -3 \sum_{i=0}^{27} \binom{27}{i} 9^i,$$

which implies that

$$\sum_{i=0}^{27} \binom{27}{i} (-3)^{2i+1} = -3 \cdot 10^{27}.$$

(2) Use the formula

$$\Big(\sum_{i=0}^{n} a_i x^i\Big)\Big(\sum_{j=0}^{m} b_j x^j\Big) = \sum_{r=0}^{n+m} \Big(\sum_{k=0}^{r} a_k b_{r-k}\Big) x^r$$

where it is understood that $a_i = 0$ for $i > n$ and $b_j = 0$ for $j > m$. The binomial theorem says that

$$(1+x)^{m+n} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^k.$$

Put these formulas together

$$\begin{aligned}
\sum_{k=0}^{n+m} \binom{n+m}{k} x^k = (1+x)^{m+n} &= (1+x)^m (1+x)^n \\
&= \Big(\sum_{k=0}^{m} \binom{m}{k} x^k\Big)\Big(\sum_{r=0}^{n} \binom{n}{r} x^r\Big) \\
&= \sum_{r=0}^{n+m} \Big(\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}\Big) x^r
\end{aligned}$$

Now compare coefficients of x^j, for $j = 1, \dots, n+m$, which gives (1).

(3) Set $a = b = r = n$, then Vandermonde's formula says that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}.$$

With

$$\binom{n}{k} = \binom{n}{n-k}$$

the statement follows.

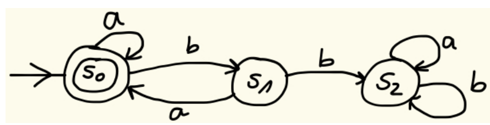
Exercise 6. Finite state automata (10 points)

(1) **(7 points)** *Draw the state diagram $D(M)$ of the automaton M with states $S := \{s_0, s_1, s_2\}$, accepting states $Y := \{s_0\}$, input alphabet $I := \{a, b\}$, described in the state table $T(M)$:*

	ν	
	a	b
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_2	s_2

(2) **(3 points)** *Write a regular expression for the language accepted by M.*

Solution 6. (1)



(2) $(a \vee ba)^*$

Exercise 7. Graphs (10 points)

(1) **(5 points)** *Is there an undirected graph with 102 vertices, such that exactly 49 vertices have degree 5, and the remaining 53 vertices have degree 6?*

(2) **(5 points)** *If a planar graph has 12 vertices, each of degree 3, how many regions and edges does the graph have?*

Solution 7. (1) Graph $G = (V, E)$: use formula $2|E| = \sum_{v \in V} \deg(v)$, which says that

$$49 \cdot 5 + 53 \cdot 6 = 2|E|.$$

The righthand side is an even number, whereas the lefthand side is an odd number. Hence, there is no such graph.

(2) Graph $G = (V, E)$: use formula $2|E| = \sum_{v \in V} \deg(v)$, which gives the number of edges, $|E| = 18$. Then use Euler's formula $|V| - |E| + |R| = 2$, which gives the number of regions, $|R| = 8$.